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Limit lamination theorems for H-surfaces

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Abstract

In this paper we prove some general results for constant mean curvature lamination limits of certain sequences of compact surfaces M_n embedded in \mathbb{R}^3 with constant mean curvature H_n and fixed finite genus, when the boundaries of these surfaces tend to infinity. Two of these theorems generalize to the non-zero constant mean curvature case, similar structure theorems by Colding and Minicozzi in [6, 8] for limits of sequences of minimal surfaces of fixed finite genus.

Mathematics Subject Classification: Primary 53A10, Secondary 49Q05, 53C42

Key words and phrases: Minimal surface, constant mean curvature, one-sided curvature estimate, curvature estimates, minimal lamination, H -surface, H -lamination, chord-arc, removable singularity, positive injectivity radius.

1 Introduction

In this paper we apply results in [28, 29, 30, 31] to obtain (after passing to a subsequence) constant mean curvature lamination limits for sequences of compact surfaces M_n embedded in \mathbb{R}^3 with constant mean curvature H_n and fixed finite genus, when the boundaries of these surfaces tend to infinity in \mathbb{R}^3 . These lamination limit results are inspired by and generalize to the non-zero constant mean curvature setting similar structure theorems by Colding and Minicozzi in [6, 8] in the case of embedded minimal surfaces; also see some closely related work of Meeks, Perez and Ros in [19, 21] in the minimal setting.

For clarity of exposition, we will call an oriented surface M immersed in \mathbb{R}^3 an H -surface if it is *embedded* and it has *non-negative constant mean curvature* H . In this manuscript $\mathbb{B}(R)$ denotes the open ball in \mathbb{R}^3 centered at the origin $\vec{0}$ of

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radius R and for a point p on a surface Σ in \mathbb{R}^3 , $|A_\Sigma|(p)$ denotes the norm of the second fundamental form of Σ at p .

Definition 1.1. Let U be an open set in \mathbb{R}^3 .

1. We say that a sequence of smooth surfaces $\Sigma(n) \subset U$ has *locally bounded norm of the second fundamental form in U* if for every compact subset B in U , the norms of the second fundamental forms of the surfaces $\Sigma(n)$ are uniformly bounded in B .
2. We say that a sequence of smooth surfaces $\Sigma(n) \subset U$ has *locally positive injectivity radius in U* if for every compact subset B in U , the injectivity radius functions of the surfaces $\Sigma(n)$ at points in B are bounded away from zero for n sufficiently large; see Definition 2.13 for the definition of the injectivity radius function.
3. We say that a sequence of smooth surfaces $\Sigma(n) \subset U$ has *uniformly positive injectivity radius in U* if there exists an $\varepsilon > 0$ such that for every compact subset B in U , the injectivity radius functions of the surfaces $\Sigma(n)$ at points in B are bounded from below by ε for n sufficiently large.

We will also need the next definition in the statement of Theorems 1.3 below, as well as Definition 2.12 of the flux of a 1-cycle in an H -surface, in the statements of Theorems 1.3 and 1.5 below.

Definition 1.2. A *strongly Alexandrov embedded H -surface* $f: \Sigma \rightarrow \mathbb{R}^3$ is a proper immersion of a complete surface Σ of constant mean curvature H that extends to a proper immersion of a complete three-manifold W so that Σ is the mean convex boundary of W and $f|_{\text{Int}(W)}$ is injective. See [33] for further discussion on this notion.

In this paper we wish to describe for any large radius $R > 0$, the geometry in $\mathbb{B}(R)$ of any connected compact H -surface M in \mathbb{R}^3 of fixed finite genus that passes through the origin and satisfies:

1. the non-empty boundary of M lies much farther than R from the origin;
2. the injectivity radius function of M is not too small at points in $\mathbb{B}(R)$.

In order to obtain this geometric description of M , it is natural to consider a sequence $\{M_n\}_{n \in \mathbb{N}}$ of compact H_n -surfaces in \mathbb{R}^3 with finite genus at most k , $\vec{0} \in M_n$, M_n contains no spherical components, $\partial M_n \subset [\mathbb{R}^3 - \mathbb{B}(n)]$ and such that the sequence has locally bounded injectivity radius in \mathbb{R}^3 . Then after passing

to a subsequence and possibly translating the surfaces M_n by vectors of uniformly bounded length so that $\vec{0} \in M_n$ still holds, then exactly one of the following three possibilities occurs in the sequence:

1. $\{M_n\}_{n \in \mathbb{N}}$ has locally bounded norm of the second fundamental form in \mathbb{R}^3 .
2. $\lim_{n \rightarrow \infty} |A_{M_n}|(\vec{0}) = \infty$ and $\lim_{n \rightarrow \infty} I_{M_n}(\vec{0}) = \infty$.
3. $\lim_{n \rightarrow \infty} |A_{M_n}|(\vec{0}) = \infty$ and $\lim_{n \rightarrow \infty} I_{M_n}(\vec{0}) = C$, for some $C > 0$

Depending on which of the above three mutually exclusive conditions holds for $\{M_n\}_{n \in \mathbb{N}}$, one has a limit geometric description given by its corresponding theorem listed below.

The next theorem corresponds to the case where $\{M_n\}_{n \in \mathbb{N}}$ has locally bounded norm of the second fundamental form in \mathbb{R}^3 .

Theorem 1.3. *Suppose that $\{M_n\}_{n \in \mathbb{N}}$ is a sequence of compact H_n -surfaces in \mathbb{R}^3 with finite genus at most k , $\vec{0} \in M_n$, M_n contains no spherical components, $\partial M_n \subset [\mathbb{R}^3 - \mathbb{B}(n)]$ and the sequence has locally bounded norm of the second fundamental form in \mathbb{R}^3 . Then, after replacing $\{M_n\}_{n \in \mathbb{N}}$ by a subsequence, the sequence of surfaces $\{M_n\}_{n \in \mathbb{N}}$ converges with respect to the C^α -norm, for any $\alpha \in (0, 1)$, to a minimal lamination M_∞ of \mathbb{R}^3 by parallel planes or it converges smoothly (with multiplicity one or two) to a possibly disconnected, strongly Alexandrov embedded H -surface M_∞ of genus at most k and every component of M_∞ is non-compact. Moreover:*

1. *If the convergent sequence has uniformly positive injectivity radius in \mathbb{R}^3 or if $H = 0$, then the norm of the second fundamental form of M_∞ is bounded.*
2. *If there exist positive numbers I_0, H_0 such that for n large either the injectivity radius functions of the M_n at $\vec{0}$ are bounded from above by I_0 or $H_n \geq H_0$, then the limit object is a possibly disconnected, strongly Alexandrov embedded H -surface M_∞ and there exist a positive constant $\eta = \eta(M_\infty)$ and simple closed oriented curves $\gamma_n \subset M_n$ with scalar fluxes $F(\gamma_n)$ with $\lim_{n \rightarrow \infty} F(\gamma_n) = \eta$.*

The next theorem corresponds to the case where $\lim_{n \rightarrow \infty} |A_{M_n}|(\vec{0}) = \infty$ and $\lim_{n \rightarrow \infty} I_{M_n}(\vec{0}) = \infty$.

Theorem 1.4. *Suppose that $\{M_n\}_{n \in \mathbb{N}}$ is a sequence of compact H_n -surfaces in \mathbb{R}^3 with finite genus at most k , $\vec{0} \in M_n$, M_n contains no spherical components, $\partial M_n \subset [\mathbb{R}^3 - \mathbb{B}(n)]$, the sequence has locally positive injectivity radius in \mathbb{R}^3 , $\lim_{n \rightarrow \infty} |A_{M_n}|(\vec{0}) = \infty$ and $\lim_{n \rightarrow \infty} I_{M_n}(\vec{0}) = \infty$.*

Let $\mathcal{S} \subset \mathbb{R}^3$ denote the x_3 -axis. Then, after replacing by a subsequence and applying a fixed rotation that fixes the origin:

1. $\{M_n\}_{n \in \mathbb{N}}$ converges with respect to the C^α -norm, for any $\alpha \in (0, 1)$, to the minimal foliation \mathcal{L} of $\mathbb{R}^3 - \mathcal{S}$ by horizontal planes punctured at points in \mathcal{S} .
2. For any $R > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$, there exists a possibly disconnected compact subdomain \mathcal{C}_n of M_n , with $[M_n \cap \mathbb{B}(R/2)] \subset \mathcal{C}_n \subset \mathbb{B}(R)$ and with $\partial \mathcal{C}_n \subset \mathbb{B}(R) - \mathbb{B}(R/2)$, consisting of a disk \mathcal{D}_n containing the origin and possibly a second disk that intersects $\mathbb{B}(R/n)$, where each disk has intrinsic diameter less than $3R$.
3. Away from \mathcal{S} , each component of \mathcal{C}_n consists of two multi-valued graphs spiraling together to form a double spiral staircase (see Remark 2.11 for an explicit geometric description of the double spiral staircase structure for \mathcal{C}_n).

The last theorem corresponds to the case where $\lim_{n \rightarrow \infty} |A_{M_n}|(\vec{0}) = \infty$ and $\lim_{n \rightarrow \infty} I_{M_n}(\vec{0}) = C$.

Theorem 1.5. Suppose that $\{M_n\}_{n \in \mathbb{N}}$ is a sequence of compact H_n -surfaces in \mathbb{R}^3 with finite genus at most k , $\vec{0} \in M_n$, M_n contains no spherical components, $\partial M_n \subset [\mathbb{R}^3 - \mathbb{B}(n)]$, the sequence has locally positive injectivity radius in \mathbb{R}^3 , $\lim_{n \rightarrow \infty} |A_{M_n}|(\vec{0}) = \infty$ and $\lim_{n \rightarrow \infty} I_{M_n}(\vec{0}) = C$, for some $C > 0$.

Let $\mathcal{S}_0 = \{(0, 0, t) \mid t \in \mathbb{R}\}$, $\mathcal{S}_C = \{(C, 0, t) \mid t \in \mathbb{R}\}$ and $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_C$. Then, after replacing by a subsequence and applying a fixed rotation that fixes the origin:

1. $\{M_n\}_{n \in \mathbb{N}}$ converges with respect to the C^α -norm, for any $\alpha \in (0, 1)$, to the minimal foliation \mathcal{L} of $\mathbb{R}^3 - \mathcal{S}$ by horizontal planes punctured at points in \mathcal{S} .
2. Given $R > C$ there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$, the subdomain Δ_n of $M_n \cap \mathbb{B}(R)$ that intersects $\mathbb{B}(\frac{R}{4})$ is a planar domain. In fact, Δ_n consists of a connected planar domain $\Delta_1(n)$ containing the origin and possibly a second connected planar domain $\Delta_2(n)$ and $\Delta_2(n) \cap \mathbb{B}(\frac{R}{n}) \neq \emptyset$. Moreover, the intrinsic distance in M_n between any two points in the same connected component of Δ_n is less than $3R$. Away from \mathcal{S} , each component of Δ_n consists of exactly two multi-valued graphs spiraling together. Near \mathcal{S}_0 and \mathcal{S}_C , the pair of multi-valued graphs form double spiral staircases with opposite handedness (see Remark 2.11 for a geometric description of each of the 1 or 2 components of Δ_n near points of \mathcal{S}). Thus, circling only \mathcal{S}_0 or only \mathcal{S}_C in Δ_n results in going either up or down, while a path circling both \mathcal{S}_0 and \mathcal{S}_C closes up.
3. There exist simple closed oriented curves $\gamma_n \subset M_n$ converging to the line segment joining the pair of points in $\mathcal{S} \cap \{x_3 = 0\}$ and having lengths converging to $2C$ and fluxes converging to $(0, 2C, 0)$.

In [29] we apply the non-zero flux conclusions in Theorems 1.3 and 1.5 to obtain curvature estimates away from the boundary for any compact 1-annulus in \mathbb{R}^3 that has scalar flux that is either zero or greater than some $\rho > 0$; see Corollary 5.4 in [29] for this result.

The geometric description in item 2 of Theorem 1.5 is identical to the geometric description of the $H = 0$ case given in Theorem 0.9 of paper [8] by Colding and Minicozzi, where in their situation the number of components in $\mathcal{C}_n(R)$ must be one. When the hypotheses of Theorem 1.4 or 1.5 hold, as n approaches infinity the convergent geometry of the surfaces M_n around the line or pair of lines in \mathcal{S} is that of a so-called “parking garage structure”. See for instance [19] for the general notion and theory of parking garage surfaces in \mathbb{R}^3 and the notion of the convergence of these surfaces to a limit “parking garage structure”. This kind of limiting structure and its application to obtain curvature estimates for certain minimal planar domains in \mathbb{R}^3 first appeared in work of Meeks, Perez and Ros in [22].

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2 Preliminaries.

Throughout this paper, we use the following notation. Given $a, b, R > 0$, $p \in \mathbb{R}^3$ and Σ a surface in \mathbb{R}^3 :

- $\mathbb{B}(p, R)$ is the open ball of radius R centered at p .
- $\mathbb{B}(R) = \mathbb{B}(\vec{0}, R)$, where $\vec{0} = (0, 0, 0)$.
- For $p \in \Sigma$, $B_\Sigma(p, R)$ denotes the open intrinsic ball in Σ of radius R .
- $A(r_1, r_2) = \{(x_1, x_2, 0) \mid r_2^2 \leq x_1^2 + x_2^2 \leq r_1^2\}$.

We first introduce the notion of multi-valued graph, see [4] for further discussion. Intuitively, an N -valued graph is a simply-connected embedded surface covering an annulus such that over a neighborhood of each point of the annulus, the surface consists of N graphs. The stereotypical infinite multi-valued graph is half of the helicoid, i.e., half of an infinite double-spiral staircase.

Definition 2.1 (Multi-valued graph). Let \mathcal{P} denote the universal cover of the punctured (x_1, x_2) -plane, $\{(x_1, x_2, 0) \mid (x_1, x_2) \neq (0, 0)\}$, with global coordinates (ρ, θ) .

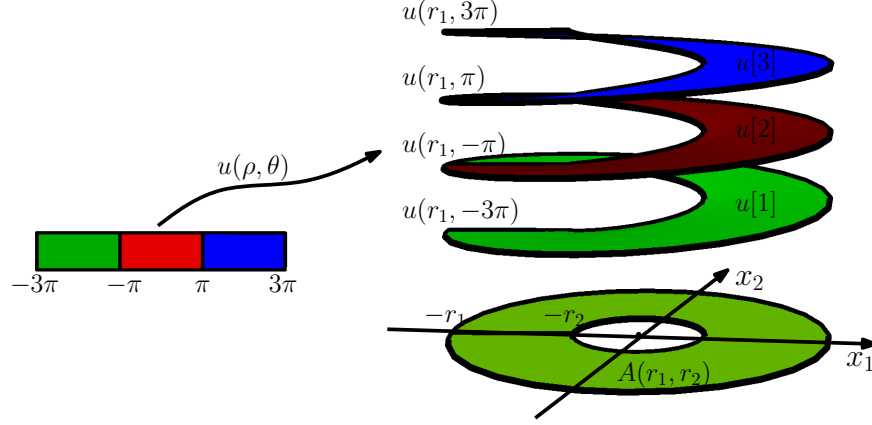


Figure 1: A right-handed 3-valued graph.

1. An N -valued graph over the annulus $A(r_1, r_2)$ is a single valued graph $u(\rho, \theta)$ over $\{(\rho, \theta) \mid r_2 \leq \rho \leq r_1, |\theta| \leq N\pi\} \subset \mathcal{P}$, if N is odd, or over $\{(\rho, \theta) \mid r_2 \leq \rho \leq r_1, (-N+1)\pi \leq \theta \leq \pi(N+1)\} \subset \mathcal{P}$, if N is even.
2. An N -valued graph $u(\rho, \theta)$ over the annulus $A(r_1, r_2)$ is called *right-handed* [*lefthanded*] if whenever it makes sense, $u(\rho, \theta) < u(\rho, \theta + 2\pi)$ [$u(\rho, \theta) > u(\rho, \theta + 2\pi)$]
3. The set $\{(r_2, \theta, u(r_2, \theta)), \theta \in [-N\pi, N\pi]\}$ when N is odd (or $\{(r_2, \theta, u(r_2, \theta)), \theta \in [(-N+1)\pi, (N+1)\pi]\}$ when N is even) is the *inner boundary* of the N -valued graph.

From Theorem 2.23 in [29] one obtains the following, detailed geometric description of an H -disk with large norm of the second fundamental form at the origin. The precise meanings of certain statements below are made clear in [29] and we refer the reader to that paper for further details.

Theorem 2.2. *Given $\varepsilon, \tau > 0$ and $\bar{\varepsilon} \in (0, \varepsilon/4)$ there exist constants $\Omega_\tau := \Omega(\tau)$, $\omega_\tau := \omega(\tau)$ and $G_\tau := G(\varepsilon, \tau, \bar{\varepsilon})$ such that if M is an H -disk, $H \in (0, \frac{1}{2\varepsilon})$, $\partial M \subset \partial \mathbb{B}(\varepsilon)$, $\vec{0} \in M$ and $|A_M|(\vec{0}) > \frac{1}{\eta} G_\tau$, for $\eta \in (0, 1]$, then for any $p \in \mathbb{B}(\vec{0}, \eta\bar{\varepsilon})$ that is a maximum of the function $|A_M|(\cdot)(\eta\bar{\varepsilon} - |\cdot|)$, after translating M by $-p$, the following geometric description of M holds:*

- *On the scale of the norm of the second fundamental form M looks like one or two helicoids nearby the origin and, after a rotation that turns these helicoids into vertical helicoids, M contains a 3-valued graph u over $A(\varepsilon/$*

$\Omega_\tau, \frac{\omega_\tau}{|A_M|(\bar{0})})$ with norm of its gradient less than τ and with inner boundary in $\mathbb{B}(10 \frac{\omega_\tau}{|A_M|(\bar{0})})$.

- Moreover, given $j \in \mathbb{N}$ if we let the constant G_τ depend on j as well, then M contains j pairwise disjoint 3-valued graphs with their inner boundaries in $\mathbb{B}(10 \frac{\omega_\tau}{|A_M|(\bar{0})})$.

Theorem 2.2 was inspired by the pioneering work of Colding and Minicozzi in the minimal case [3, 4, 5, 6]; however in the constant positive mean curvature setting this description has led to a different conclusion, that is the existence of the intrinsic curvature estimates stated below.

Theorem 2.3 (Intrinsic curvature estimates, Theorem 1.3 in [29]). *Given $\delta, \mathcal{H} > 0$, there exists a constant $K(\delta, \mathcal{H})$ such that for any H -disk \mathcal{D} with $H \geq \mathcal{H}$,*

$$\sup_{\{p \in \mathcal{D} \mid d_{\mathcal{D}}(p, \partial \mathcal{D}) \geq \delta\}} |A_{\mathcal{D}}| \leq K(\delta, \mathcal{H}).$$

Rescalings of a helicoid give a sequence of embedded minimal disks with arbitrarily large norm of the second fundamental form at points arbitrarily far from its boundary; therefore in the minimal setting, similar curvature estimates do not hold.

The next two results from [31] will also be essential tools that we use in this paper.

Theorem 2.4 (Extrinsic one-sided curvature estimates for H -disks). *There exist $\varepsilon \in (0, \frac{1}{2})$ and $C \geq 2\sqrt{2}$ such that for any $R > 0$, the following holds. Let \mathcal{D} be an H -disk such that*

$$\mathcal{D} \cap \mathbb{B}(R) \cap \{x_3 = 0\} = \emptyset \quad \text{and} \quad \partial \mathcal{D} \cap \mathbb{B}(R) \cap \{x_3 > 0\} = \emptyset.$$

Then:

$$\sup_{x \in \mathcal{D} \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\}} |A_{\mathcal{D}}|(x) \leq \frac{C}{R}. \quad (1)$$

In particular, if $\mathcal{D} \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\} \neq \emptyset$, then $H \leq \frac{C}{R}$.

The next corollary follows immediately from Theorem 2.4 by a simple rescaling argument. It roughly states that we can replace the (x_1, x_2) -plane by any surface that has a fixed uniform estimate on the norm of its second fundamental form.

Corollary 2.5. *Given an $a \geq 0$, there exist $\varepsilon \in (0, \frac{1}{2})$ and $C_a > 0$ such that for any $R > 0$, the following holds. Let Δ be a compact immersed surface in $\mathbb{B}(R)$*

with $\partial\Delta \subset \partial\mathbb{B}(R)$, $\vec{0} \in \Delta$ and satisfying $|A_\Delta| \leq a/R$. Let \mathcal{D} be an H -disk such that

$$\mathcal{D} \cap \mathbb{B}(R) \cap \Delta = \emptyset \quad \text{and} \quad \partial\mathcal{D} \cap \mathbb{B}(R) = \emptyset.$$

Then:

$$\sup_{x \in \mathcal{D} \cap \mathbb{B}(\varepsilon R)} |A_{\mathcal{D}}|(x) \leq \frac{C_a}{R}. \quad (2)$$

In particular, if $\mathcal{D} \cap \mathbb{B}(\varepsilon R) \neq \emptyset$, then $H \leq \frac{C_a}{R}$.

The next curvature estimate is a more involved application of Theorem 2.4 and also uses Theorem 2.10 below in its proof.

Corollary 2.6 (Corollary 4.6 in [30]). *There exist constants $\varepsilon < 1$, $C > 1$ such that the following holds. Let $\Sigma_1, \Sigma_2, \Sigma_3$ be three pairwise disjoint H_i -disks with $\partial\Sigma_i \subset [\mathbb{R}^3 - \mathbb{B}(1)]$ for $i = 1, 2, 3$. If $\mathbb{B}(\varepsilon) \cap \Sigma_i \neq \emptyset$ for $i = 1, 2, 3$, then*

$$\sup_{\mathbb{B}(\varepsilon) \cap \Sigma_i, i=1,2,3} |A_{\Sigma_i}| \leq C.$$

In [28], we applied the one-sided curvature estimates in Theorem 2.4 to prove a relation between intrinsic and extrinsic distances in an H -disk, which can be viewed as a *weak chord arc* property. This result was motivated by and generalizes a previous result by Colding-Minicozzi for 0-disks, namely Proposition 1.1 in [7]. We begin by making the following definition.

Definition 2.7. *Given a point p on a surface $\Sigma \subset \mathbb{R}^3$, $\Sigma(p, R)$ denotes the closure of the component of $\Sigma \cap \mathbb{B}(p, R)$ passing through p .*

Theorem 2.8 (Weak chord arc property, Theorem 1.2 in [28]). *There exists a $\delta_1 \in (0, \frac{1}{2})$ such that the following holds.*

Let Σ be an H -disk in \mathbb{R}^3 . Then for all intrinsic closed balls $\overline{B}_\Sigma(x, R)$ in $\Sigma - \partial\Sigma$:

1. $\Sigma(x, \delta_1 R)$ is a disk with piecewise smooth boundary $\partial\Sigma(x, \delta_1 R) \subset \partial\mathbb{B}(\delta_1 R)$.
2. $\Sigma(x, \delta_1 R) \subset B_\Sigma(x, \frac{R}{2})$.

For applications here, we will also need the closely related chord-arc result below, that is Theorem 1.2 in [30].

Theorem 2.9 (Chord arc property for H -disks). *There exists a constant $a > 1$ so that the following holds. Suppose that Σ is an H -disk with $\vec{0} \in \Sigma$, $R > r_0 > 0$ and $B_\Sigma(\vec{0}, aR) \subset \Sigma - \partial\Sigma$. If $\sup_{B_\Sigma(\vec{0}, (1-\frac{\sqrt{2}}{2})r_0)} |A_\Sigma| > r_0^{-1}$, then*

$$\frac{1}{3} \text{dist}_\Sigma(x, \vec{0}) \leq |x|/2 + r_0, \text{ for } x \in B_\Sigma(\vec{0}, R).$$

Since in the proofs of Theorems 1.3, 1.4 and 1.5 we will frequently refer to parts of the statement of the Limit Lamination Theorem for H -disks, namely Theorem 1.1 in [30], we state it below for more direct referencing.

Theorem 2.10 (Limit lamination theorem for H -disks). *Fix $\varepsilon > 0$ and let $\{M_n\}_n$ be a sequence of H_n -disks in \mathbb{R}^3 containing the origin and such that $\partial M_n \subset [\mathbb{R}^3 - \mathbb{B}(n)]$ and $|A_{M_n}|(\vec{0}) \geq \varepsilon$. Then, after replacing by some subsequence, exactly one of the following two statements hold.*

- A. *The surfaces M_n converge smoothly with multiplicity one or two on compact subsets of \mathbb{R}^3 to a helicoid M_∞ containing the origin. Furthermore, every component Δ of $M_n \cap \mathbb{B}(1)$ is an open disk whose closure $\bar{\Delta}$ in M_n is a compact disk with piecewise smooth boundary, and where the intrinsic distance in M_n between any two points of its closure $\bar{\Delta}$ less than 10.*
- B. *There are points $p_n \in M_n$ such that*

$$\lim_{n \rightarrow \infty} p_n = \vec{0} \text{ and } \lim_{n \rightarrow \infty} |A_{M_n}|(p_n) = \infty,$$

and the following hold:

- (a) *The surfaces M_n converge to a foliation of \mathbb{R}^3 by planes and the convergence is C^α , for any $\alpha \in (0, 1)$, away from the line containing the origin and orthogonal to the planes in the foliation.*
- (b) *There exists compact subdomains \mathcal{C}_n of M_n , $[M_n \cap \bar{\mathbb{B}}(1)] \subset \mathcal{C}_n \subset \mathbb{B}(2)$ and $\partial \mathcal{C}_n \subset \mathbb{B}(2) - \bar{\mathbb{B}}(1)$, each \mathcal{C}_n consisting of one or two pairwise disjoint disks, where each disk component has intrinsic diameter less than 3 and intersects $\mathbb{B}(1/n)$. Moreover, each connected component of $M_n \cap \mathbb{B}(1)$ is an open disk whose closure in M_n is a compact disk with piecewise smooth boundary.*

Remark 2.11 (Double spiral staircase structure). Suppose that Case B occurs in the statement of Theorem 2.10 and let Δ_n be a component of \mathcal{C}_n . By Remark 3.6 in [30], after replacing the surfaces M_n by a subsequence and composing them by a rotation of \mathbb{R}^3 that fixes the origin and so that the planes of the limit foliation are horizontal, then, as n tends to infinity, Δ_n has the structure of a *double spiral staircase*, in the following sense:

1. Δ_n contains a smooth connected arc $\Gamma_n(t)$, called its *central column*, that is parameterized by the set of its third coordinates which equals the interval $I_n = (-1 - \frac{1}{n}, 1 + \frac{1}{n})$. $\Gamma_n(t)$ is the set of points of Δ_n with vertical tangent

planes and $\Gamma_n(t)$ is $\frac{1}{n}$ -close to the arc $\{(0, 0, t) \mid t \in I_n\}$ with respect to the C^1 -norm.

For each $t \in I_n$, let $T_n(t)$ be the vertical tangent plane of Δ_n at $\Gamma_n(t)$.

2. For $t \in I_n$, $T_n(t) \cap \Delta_n$ contains a smooth arc $\alpha_{n,t}$ passing through $\Gamma_n(t)$ that is $\frac{1}{n}$ -close in the C^1 -norm to an arc $\beta_{n,t}$ of the line $T_n(t) \cap \{x_3 = t\}$ such that $\Gamma_n(t) \in \beta_{n,t}$ and the end points of $\beta_{n,t}$ lie in $\mathbb{B}(2) - \overline{\mathbb{B}}(1)$; here $\{\alpha_{n,t}\}_{t \in I_n}$ is a pairwise disjoint collection of arcs and $\Delta_n = \bigcup_{t \in I_n} \alpha_{n,t}$.
3. The absolute Gaussian curvature of Δ_n along $\Gamma_n(t)$ is pointwise greater than n . Since the central column $\Gamma_n(t)$ of Δ_n is converging C^1 to the segment given by $\mathbb{B}(1) \cap \{x_3\text{-axis}\}$, the arcs $\alpha_{n,t}$ are converging to $T_n(t) \cap \Delta_n$ and on the scale of curvature Δ_n is closely approximated by a vertical helicoid near every point of $\Gamma_n(t)$ (see Corollary 3.8 in [31]), then the rate of change of the horizontal unit normal of $T_n(t)$ along $\Gamma_n(t)$ is greater than \sqrt{n} .

Next, we recall the notion of flux of a 1-cycle of an H -surface; see for instance [15, 16, 36] for further discussions of this invariant.

Definition 2.12. Let γ be a 1-cycle in an H -surface M . The *flux* of γ is $F(\gamma) = \int_\gamma (H\gamma + \xi) \times \dot{\gamma}$, where ξ is the unit normal to M along γ . The norm $|F(\gamma)|$ is called the *scalar flux* of γ .

The flux of a 1-cycle in an H -surface M is a homological invariant and we say that M has *zero flux* if the flux of any 1-cycle in M is zero; in particular, since the first homology group of a disk is zero, an H -disk has zero flux. Finally, the next definition was needed in Definition 1.1 in the Introduction.

Definition 2.13. The injectivity radius $I_M(p)$ at a point p of a complete Riemannian manifold M is the supremum of the radii $r > 0$ of the open metric balls $B_M(p, r)$ for which the exponential map at p is a diffeomorphism. This defines the *injectivity radius function*, $I_M: M \rightarrow (0, \infty]$, which is continuous on M (see e.g., Proposition 88 in [1]). When M is complete, we let $\text{Inj}(M)$ denote the *injectivity radius of M* , which is defined to be the infimum of I_M .

3 The proof of Theorem 1.3.

In this section we will prove all of the statements in Theorem 1.3 except for one of the implications in item 2. However, at the end of this section we explain how the missing proof of this implication follows from item 2 of Theorem 1.5. Hence, once Theorem 1.5 is proven in Section 5, the proof of Theorem 1.3 will be complete.

Let $\{M_n\}_{n \in \mathbb{N}}$ be a sequence of compact H_n -surfaces in \mathbb{R}^3 with finite genus at most k , $\vec{0} \in M_n$, M_n contains no spherical components, $\partial M_n \subset [\mathbb{R}^3 - \mathbb{B}(n)]$ and the sequence has locally bounded norm of the second fundamental form in \mathbb{R}^3 . By a standard argument, a subsequence of the surfaces converges to a weak H -lamination \mathcal{L} of \mathbb{R}^3 ; see the references [20, 25, 32] for this argument and the Appendix for the definition and some key properties of a weak H -lamination that we will apply below.

Let L be a leaf of \mathcal{L} . If L is stable (L admits a positive Jacobi function), then L is a complete, stable constant mean curvature surface in \mathbb{R}^3 , which must be a flat plane by [17, 35]. If L is a flat plane, then the injectivity radius is infinite. Since by Theorem 4.3 in [24] limit leaves (see Definition 6.3 for the definition of limit leaf) of \mathcal{L} are stable, we conclude that if L is a limit leaf of \mathcal{L} , then it is a plane and has infinite injectivity radius. Thus we also conclude that if \mathcal{L} has a limit leaf, then $H = 0$. From this point till the beginning of the proof of item 1 of the theorem, we will assume that \mathcal{L} is not a lamination of \mathbb{R}^3 by parallel planes.

Suppose now that L is a non-flat leaf of \mathcal{L} . By the discussion in the previous paragraph and item 3 of Remark 6.4, L is a non-limit leaf and it has on its mean convex side an embedded half-open regular neighborhood $N(L)$ in \mathbb{R}^3 that intersects \mathcal{L} only in the leaf L ; also since L is not a limit leaf of \mathcal{L} , then $N(L)$ lies in the interior of an open set $\widehat{N}(L)$ that also intersects \mathcal{L} only in the leaf L . Since the leaf L is not stable, the existence of $N(L)$ allows us to apply the arguments in the proof of Case A in the proof of Proposition 3.1 of [31], to show that the sequence $\{M_n \cap \widehat{N}(L)\}_{n \in \mathbb{N}}$ converges to L with multiplicity one or two and the genus of L is at most k .

We now prove that \mathcal{L} does not contain a limit leaf. Arguing by contradiction, suppose L is a limit leaf of \mathcal{L} , then, as previously proved, L must be a flat plane. Thus, since we are assuming that \mathcal{L} is not a lamination of \mathbb{R}^3 by parallel planes, \mathcal{L} is a minimal lamination containing a flat leaf L and a non-flat leaf L' with finite genus at most k . By Theorem 7 in [27], a finite genus leaf of a minimal lamination of \mathbb{R}^3 is proper, which contradicts the Half-space Theorem [14] since L' is contained in the half-space determined by L . This proves that \mathcal{L} contains no limit leaves.

Since \mathcal{L} is a weak H -lamination of \mathbb{R}^3 that does not have a limit leaf, then the union of the leaves of \mathcal{L} is a properly immersed, possibly disconnected H -surface, such that around any point p where the leaves of the weak lamination do not form a lamination, there exists an $\varepsilon > 0$ such that $\mathcal{L} \cap \mathbb{B}(p, \varepsilon)$ consists of exactly two disks in leaves of \mathcal{L} with boundaries in $\mathbb{B}(p, \varepsilon)$ and these two disks lie on one side of each other, intersect at p and their non-zero mean curvature vectors are oppositely oriented. See the Appendix for further discussion of properties of weak H -laminations.

If $H = 0$, then the leaves of \mathcal{L} are embedded by the maximum principle and \mathcal{L}

is connected because of the Strong Halfspace Theorem [14]. Thus by elementary separation properties, \mathcal{L} bounds a proper region W of \mathbb{R}^3 . Hence, \mathcal{L} is a connected, strongly Alexandrov embedded minimal surface in the case where $H = 0$.

Suppose next that $H > 0$ and note that by the previous description or by item 3 in Remark 6.4, each leaf L of \mathcal{L} can be perturbed slightly on its mean convex side to be properly embedded and hence L is strongly Alexandrov embedded. By Theorem 2 in [12], for any two components Σ_1 and Σ_2 of \mathcal{L} , Σ_1 does not lie in the mean convex component of $\mathbb{R}^3 - \Sigma_2$. It follows that each of the components of $\mathbb{R}^3 - \mathcal{L}$, except for one, is a mean convex domain with one boundary component. This means \mathcal{L} corresponds to a possibly disconnected strongly Alexandrov embedded H -surface. Finally, since closed Alexandrov embedded H -surfaces in \mathbb{R}^3 are round spheres and no component of M_n is spherical, a monodromy argument implies that each leaf of the limit lamination \mathcal{L} is non-compact. Setting $M_\infty := \mathcal{L}$ finishes the proof of the first statement of the theorem.

We next prove item 1 in the theorem. Namely, we will prove that if the sequence $\{M_n\}_{n \in \mathbb{N}}$ has uniformly positive injectivity radius in \mathbb{R}^3 or if $H = 0$, then the norm of the second fundamental form of M_∞ is bounded. If the constant mean curvature of M_∞ is positive and if the sequence $\{M_n\}_{n \in \mathbb{N}}$ has uniformly positive injectivity radius in \mathbb{R}^3 , then the norms of the second fundamental forms of the surfaces M_n converging to M_∞ on any compact region of \mathbb{R}^3 are eventually bounded from above by a constant that only depends on the curvature estimate given in Theorem 2.3; hence M_∞ has uniformly bounded norm of its second fundamental form in this case. If the mean curvature of M_∞ is zero, then as observed already either M_∞ is a lamination of \mathbb{R}^3 by parallel planes or else M_∞ is a properly embedded connected minimal surface in \mathbb{R}^3 of finite genus. If M_∞ is a lamination of \mathbb{R}^3 by parallel planes then the claim is clearly true. Otherwise, by the classification of the asymptotic behavior of properly embedded minimal surfaces in \mathbb{R}^3 with finite genus given in the papers [2, 9, 26], the norm of the second fundamental form of the unique leaf of M_∞ is also bounded in this case. This last observation completes the proof of item 1 in the theorem.

We next consider the proof of item 2 in the theorem. Namely, we will prove that if there exist positive numbers I_0, H_0 such that for n large either the injectivity radius functions of the surfaces M_n at $\vec{0}$ are bounded from above by I_0 or $H_n \geq H_0$, then M_∞ is a strongly Alexandrov embedded H -surface and there exist a positive constant $\eta = \eta(M_\infty)$ and simple closed oriented curves $\gamma_n \subset M_n$ with scalar fluxes $F(\gamma_n)$ with $\lim_{n \rightarrow \infty} F(\gamma_n) = \eta$.

We first show that M_∞ cannot be a lamination of \mathbb{R}^3 by parallel planes. Arguing by contradiction, suppose that the sequence $\{M_n\}_{n \in \mathbb{N}}$ converges to a lamination of \mathbb{R}^3 by parallel planes. In particular $\lim_{n \rightarrow \infty} H_n = 0$ and the injectivity radius functions of the surfaces M_n at $\vec{0}$ are bounded from above by I_0 . Then, for

n large, the Gauss equation implies that the $\limsup K_{M_n}$ of the Gaussian curvature functions of the surfaces M_n is non-positive. Classical results on Jacobi fields along geodesics in such surfaces imply that for n large the exponential map of M_n at $\vec{0}$ on the closed disk in $T_{\vec{0}}M_n$ of a certain radius $r_n \in (0, I_0]$ is a local diffeomorphism that is injective on the interior of the disk but it is not injective along its boundary circle of radius r_n (see, for instance, Proposition 2.12, Chapter 13 of [10]). Moreover, since the sequence $\{M_n\}_{n \in \mathbb{N}}$ has locally bounded norm of the second fundamental form, the sequence of numbers r_n is bounded away from zero. Hence, there exists a sequence of simple closed geodesic loops $\alpha_n \subset M_n$ based at $\vec{0}$ and of lengths uniformly bounded from below and above that are smooth everywhere except possibly at $\vec{0}$. By the nature of the convergence, α_n converges to a geodesic loop in M_∞ based at $\vec{0}$. Therefore M_∞ cannot be a lamination of \mathbb{R}^3 by parallel planes.

We next prove the existence of the 1-cycles $\gamma_n \subset M_n$ with non-zero flux described in item 2 of the theorem. Since M_∞ cannot be a lamination of \mathbb{R}^3 by parallel planes, by the already proved first main statement of the theorem, the sequence $\{M_n\}_{n \in \mathbb{N}}$ converges with multiplicity one or two to a possibly disconnected, non-flat strongly Alexandrov embedded H -surface M_∞ of genus at most k . Since the convergence to M_∞ is with multiplicity one or two, a curve lifting argument shows that in order to prove that item 2 holds, it suffices to show that M_∞ has non-zero flux.

If $\lim_{n \rightarrow \infty} H_n = 0$ but the injectivity radius functions of the M_n at $\vec{0}$ are bounded from above by I_0 , then the same arguments as before imply that M_∞ is not simply-connected because a simply-connected minimal surface cannot contain a geodesic loop. Thus, by the results in [34], the finite genus minimal surface M_∞ must have non-zero flux.

It remains to consider the case that $H_n \geq H_0 > 0$. In this case M_∞ is a proper collection of non-zero constant mean curvature surfaces, each component of which is non-compact and the entire surface has finite genus at most k . Abusing the notation, let M_∞ denote the component containing the origin. If M_∞ has injectivity radius function uniformly bounded from below by a positive constant, then it has uniformly bounded norm of the second fundamental form by Theorem 2.3 and again, by the results in [34], M_∞ has non-zero flux.

In other words, item 2 can only fail if M_∞ has positive mean curvature but the injectivity radius function is not bounded from below by a positive constant. In this case we can apply the blow-up argument described in Proposition 5.9. Such a blow-up argument gives the following. Let $p_n \in M_\infty$ be a sequence of points such that $\lim_{n \rightarrow \infty} I_{M_\infty}(p_n) = 0$ and let q_n be a sequence of points with almost-minimal injectivity radius for $B_{M_\infty}(p_n, 1)$, see Definition 5.8. Then, by Proposition 5.9 there exist positive numbers R_n , $\lim_{n \rightarrow \infty} R_n = \infty$, such that after replacing by a

subsequence the component \mathbf{M}_n of $\frac{1}{I_{M_\infty}(q_n)}[M_\infty - q_n] \cap \mathbb{B}(R_n)$ containing $\vec{0}$ has boundary in $\partial\mathbb{B}(R_n)$ and the following properties hold:

- \mathbf{M}_n has finite genus at most k .
- $I_{\mathbf{M}_n}(x) \geq 1/2$ for any $x \in \mathbf{M}_n \cap \mathbb{B}(R_n/2)$ and $I_{\mathbf{M}_n}(\vec{0}) = 1$.
- The mean curvatures \mathbf{H}_n of the \mathbf{M}_n converge to zero as n goes to infinity.

Suppose for the moment that the sequence \mathbf{M}_n has locally bounded norm of the second fundamental form in \mathbb{R}^3 . By the already proven main statement of Theorem 1.3 applied to the sequence \mathbf{M}_n , a subsequence converges to a possibly disconnected, strongly Alexandrov embedded H -surface \mathbf{M}_∞ of genus at most k and every component of M_∞ is non-compact. Since $\lim_{n \rightarrow \infty} \mathbf{H}_n = 0$ but the injectivity radius functions of the \mathbf{M}_n at $\vec{0}$ are bounded from above by 1, then our previous arguments imply that the finite genus minimal surface \mathbf{M}_∞ must have non-zero flux. Hence, we may assume that the sequence \mathbf{M}_n fails to have locally bounded norm of the second fundamental form. Assume for the moment that Theorem 1.5 holds. In the case that we are considering, we can apply item 2 of Theorem 1.5 to conclude that the surfaces \mathbf{M}_n have non-zero flux, which would mean that M_∞ has non-zero flux as well. The construction of the closed curves, called connection loops, with non-zero flux is described in detail after Remark 5.6.

In summary, the proof of the Theorem 1.3 will be complete once Theorem 1.5 is proven in Section 5.

4 The proof of Theorem 1.4.

In this section we will prove Theorem 1.4. Suppose that $\{M_n\}_{n \in \mathbb{N}}$ is a sequence of compact H_n -surfaces in \mathbb{R}^3 with finite genus at most k , $\vec{0} \in M_n$, M_n contains no spherical components, $\partial M_n \subset [\mathbb{R}^3 - \mathbb{B}(n)]$, the sequence has locally positive injectivity radius in \mathbb{R}^3 and $\lim_{n \rightarrow \infty} |A_{M_n}|(\vec{0}) = \infty$. Since we will use some of the results proved here in the proof of Theorem 1.5, we will for the moment not invoke the additional hypothesis $\lim_{n \rightarrow \infty} I_{M_n}(\vec{0}) = \infty$.

Since $\lim_{n \rightarrow \infty} |A_{M_n}|(\vec{0}) = \infty$ and $I_{M_n}(\vec{0})$ is bounded from below by some positive number, then Theorem 2.3 implies that $\lim_{n \rightarrow \infty} H_n = 0$. After replacing by a subsequence, there exists a smallest closed nonempty set $\mathcal{S} \subset \mathbb{R}^3$ such that the sequence $\{M\}_{n \in \mathbb{N}}$ has locally bounded norm of the second fundamental form in $\mathbb{R}^3 - \mathcal{S}$ and converges with respect to the C^α -norm, for any $\alpha \in (0, 1)$, to a nonempty minimal lamination \mathcal{L} of $\mathbb{R}^3 - \mathcal{S}$; the set \mathcal{S} is smallest in the sense that every subsequence fails to converge to a minimal lamination in a proper subset of \mathcal{S} . The proofs of the existence of \mathcal{S} and \mathcal{L} are the same as those appearing in the

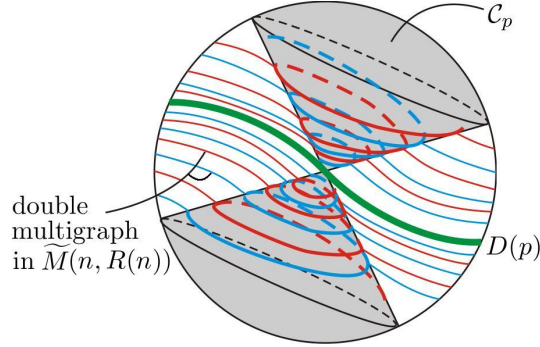


Figure 2: The local *Colding-Minicozzi picture* around a point where the curvature blows up. The stable punctured disk $D(p, *)$ appears in the limit lamination.

proofs of the first three items in Claim 3.4 in [31] and we refer the reader to [31] for the details.

We begin by studying the geometry of \mathcal{L} and \mathcal{S} . The local analysis presented here is analogous to and inspired by the one given in the minimal case considered by Colding and Minicozzi in [8]. Indeed the structure of the lamination nearby points in \mathcal{S} is identical.

Let $p \in \mathcal{S}$. After replacing by a subsequence, there exists a sequence of points $p_n \in M_n$ converging to p such that the norm of the second fundamental form of M_n at p_n is at least n . Since we may assume that the injectivity radius function of M_n is at least some $\varepsilon > 0$ at p_n , then applying Theorem 2.8, we find that for n large, the intersection of each H_n -disk $B_{M_n}(p_n, \varepsilon)$ with $\mathbb{B}(p, \delta_1 \varepsilon)$ contains a component $M_n(p_n, \delta_1 \varepsilon)$ that is an H_n -disk with boundary in the boundary of $\mathbb{B}(p_n, \delta_1 \varepsilon)$. Theorem 2.2 now gives that for n large, there exists a collection of 3-valued graphs $\{G_1(n), \dots, G_{k(n)}(n)\}$ with inner boundaries converging to p , norms of their gradients at most one and $\lim_{n \rightarrow \infty} k(n) = \infty$. Since $\{G_1(n), \dots, G_{k(n)}(n)\}$ is a collection of embedded and pair-wise disjoint 3-valued graphs contained in a compact ball, it must contain a sequence of 3-valued graphs for which the distance between the sheets is going to zero. Hence, after reindexing, we can assume that the 3-valued graphs $G_1(n)$ are collapsing in the limit to a minimal disk $D(p) \subset \mathbb{B}(p, s)$ of gradient at most one over its tangent plane at p and where $\partial D(p) \subset \partial \mathbb{B}(p, s)$ and $s < \delta_1 \varepsilon$ is fixed and depending on ε ; actually one produces the punctured graphical disk

$$D(p, *) = D(p) - \{p\}$$

as a limit and then p is seen to be a removable singularity.

By the one-sided curvature estimate in Corollary 2.5, the 3-valued graph $G_1(n)$ gives rise to curvature estimates at points of M_n nearby $G_1(n)$ and, as n goes to infinity, these curvature estimates give rise to curvature estimates in compact subsets of the complement set

$$W(p) = \mathbb{B}(p, s) - \mathcal{C}_p$$

of some closed solid double cone \mathcal{C}_p with axis being the normal line to $D(p)$ at p . In other words, after replacing by a subsequence, the surfaces M_n have locally bounded norm of the second fundamental form in $W(p)$, which implies $W(p) \cap \mathcal{S} = \emptyset$. This observation implies that for every point $p \in \mathcal{S}$, one has a minimal lamination $\mathcal{L}_{W(p)} = \mathcal{L} \cap W(p)$ of $W(p)$ as described in previous paragraphs: see Figure 2. This local picture is exactly the same as the one that occurs in the case where the mean curvatures of the surfaces M_n are zero; as in the minimal case, we refer to it as the local *Colding-Minicozzi picture* near points in \mathcal{S} . See the discussion following Definition 4.9 in [19] for a more detailed analysis of this picture.

Definition 4.1. *Given $p \in \mathcal{S}$, let L_p be the leaf of \mathcal{L} containing the punctured disk $D(p, *)$.*

The arguments appearing in the proof of this claim are based on the proof of the similar Lemmas 4.10, 4.11 and 4.12 in [19].

Claim 4.2. *The closure of L_p in \mathbb{R}^3 is a plane \overline{L}_p which intersects \mathcal{S} in a discrete set of points.*

Proof. We claim that the punctured disk $D(p, *)$ in the Colding-Minicozzi picture at p is a limit leaf of the local lamination $\mathcal{L}_W(p)$ of $W(p)$. Recall that in $W(p)$ the surfaces M_n have uniformly bounded norm of the second fundamental form at points of intersection with the annulus $A = W(p) \cap \partial \mathbb{B}(p, \varepsilon)$. After replacing \mathcal{C}_p by a cone of wider aperture and choosing $\varepsilon > 0$ sufficiently small, for n sufficiently large, the annulus A contains a pair of spiraling arcs $\alpha_1(n), \alpha_2(n) \subset A \cap M_n$ that begin at one of the boundary components of A and end at its other boundary component. Furthermore, as $n \rightarrow \infty$, the arcs $\alpha_1(n), \alpha_2(n)$ converge to a limit lamination \mathcal{L}_A of A that contains the simple closed curve $D(p, *) \cap A$, which is a graph of small gradient over its projection to the tangent space of $D(p)$ at p . Since every homotopically non-trivial simple closed curve in A intersects $\alpha_1(n)$, then by compactness, such a simple closed curve also intersects \mathcal{L}_A . In particular, $D(p, *) \cap A$ must be a limit leaf of \mathcal{L}_A . It follows that $D(p, *)$ is a limit leaf of $\mathcal{L}_W(p)$ of $W(p)$, which proves our claim.

Since the punctured disk $D(p, *)$ in the Colding-Minicozzi picture at p is a limit leaf of the local lamination $\mathcal{L}_W(p)$ of $W(p)$, L_p is a limit leaf of \mathcal{L} , and thus it is stable. Consider L_p to be a Riemannian surface with its related metric space structure, namely, the distance between two points in L_p is the infimum of the lengths of arcs on the leaf that join the two points. Let \widehat{L}_p be the abstract metric completion of L_p . Since L_p is a subset of \mathbb{R}^3 , \mathbb{R}^3 is complete and extrinsic distances are at most equal to intrinsic distances, then the inclusion map of L_p into \mathbb{R}^3 extends uniquely to a continuous map from \widehat{L}_p into \mathbb{R}^3 , and the image of \widehat{L}_p is contained in the closure \overline{L}_p of L_p in \mathbb{R}^3 . Note that this continuous map sends a point $q \in \widehat{L}_p - L_p$ to a point of $\overline{L}_p \cap \mathcal{S}$, which with an abuse of notation we still call q . Suppose $q \in \overline{L}_p \cap \mathcal{S}$ is the induced inclusion into \mathbb{R}^3 of a point in \widehat{L}_p and let $\{q_k\}_k \in L_p$ be a Cauchy sequence converging to q . If for all $q \in \overline{L}_p \cap \mathcal{S}$, the related Cauchy sequence q_k lies in the punctured disk $D(q, *)$, then the inclusion of the completion \widehat{L}_p of L_p in \mathbb{R}^3 would be a complete minimal surface in \mathbb{R}^3 . Since \widehat{L}_p would be stable outside of a discrete set of points and since for any compact minimal surface Λ with boundary, Λ is stable if and only if Λ is punctured in a finite set of points is stable, then it follows that the minimal surface \widehat{L}_p is stable. Hence, \widehat{L}_p viewed in \mathbb{R}^3 would be a plane equal to \overline{L}_p [11, 13]. Thus, in order to show that \overline{L}_p is a plane, it suffices to show that for k large, the points q_k lie in the punctured disk $D(q, *)$.

Arguing by contradiction, suppose that for some $q \in \overline{L}_p \cap \mathcal{S}$ and k large, $q_k \notin D(q, *)$. Clearly for k large, q_k is arbitrarily close to q in \mathbb{R}^3 and in particular, $q_k \in \mathbb{B}(q, s)$. Then it follows that, after extracting a subsequence, for k large, the points q_k lie in the same component Δ of $\mathbb{B}(q, s) - D(q)$. First consider the special case where q is an isolated point in $\mathcal{S} \cap \overline{\Delta}$, that is, for ρ sufficiently small, $\overline{\Delta} \cap \mathcal{S} \cap \mathbb{B}(q, \rho) = \{q\}$. Then $\overline{L}_p \cap \overline{\Delta} \cap [\mathbb{B}(q, \rho) - q]$ is a minimal lamination of $\mathbb{B}(q, \rho) - \{q\}$ with stable leaves and thus q is a removable singularity of this minimal lamination by Theorem 1.2 in [20]. This regularity property implies that for ρ sufficiently small, $\overline{\Delta} \cap \{L_p \cup \{q\}\} \cap \mathbb{B}(q, \rho)$ contains a collection of disks $\{D_n\}_{n \in \mathbb{N}}$ in \mathcal{L} with boundary curves in $\partial \mathbb{B}(q, \rho)$ that converge C^1 to the disk $D(q)$ as n goes to infinity. Since the points q_k lie in components of $\overline{\Delta} \cap \{L_p \cup \{q\}\}$ that are different from $D(q)$, then their intrinsic distances to q in \widehat{L}_p would be bounded uniformly from below by $\rho/2$ for k large; this is because each such point q_k is separated in $\mathbb{B}(q, \rho)$ from q by the disk D_n for n sufficiently large. Therefore, in this case the sequence of points q_k cannot be a Cauchy sequence converging to q in \widehat{L}_p .

The case when q is not an isolated point of $\mathcal{S} \cap \overline{\Delta}$ can be treated in a similar manner. If there exists a sequence of points $p_i \in \mathcal{S} \cap \overline{\Delta}$ converging to q , then for i large, through each of these points there would be a punctured disk

$[D'(p_i) - \{p_i\}] \subset L_q$ punctured at p_i with boundary in $\partial\mathbb{B}(p, \rho)$ and the disks $D'(p_i)$ converge C^1 to $D(p)$. If two points, x_1, x_2 , in $L_p \cap \Delta \cap \mathbb{B}(q, \rho/2)$, ρ small, lie on different disks or are separated in Δ by one of the disks $D'(p_i)$, then the distance between x_1 and x_2 in L_p would be bounded from below by $\rho/3$ for k large. Since the points p_i converge in $\Delta \cap \mathbb{B}(q, \rho/2)$ to q , there is always a disk $D'(p_i)$ that eventually separates points in the Cauchy sequence $\{q_k\}_k$. Hence, the sequence $\{q_k\}_k$ cannot be a Cauchy sequence unless, for k large, the points q_k lie in $D(q, *)$. By the argument given in the first paragraph of the proof, this implies that \overline{L}_p is a plane.

Finally, the fact that $\overline{L}_p \cap \mathcal{S}$ is a discrete set of points in the plane \overline{L}_p follows from the geometry of the Colding-Minicozzi picture. This completes the proof of the claim. \square

By Claim 4.2, for any $p \in \mathcal{S}$ the closure of L_p in \mathbb{R}^3 is a plane \overline{L}_p which intersects \mathcal{S} in a discrete, therefore countable, set of points. After applying a fixed rotation around the origin, we will assume that $L_{\bar{0}}$ and L_p , for any $p \in \mathcal{S}$, are horizontal planes.

The arguments already considered in the proof of Theorem 1.3 can be adapted to show that the leaves of the minimal lamination \mathcal{L} have genus at most k and hence have finite genus. For the sake of completeness, we include the proof of this key topological property for the leaves of \mathcal{L} .

Claim 4.3. *Each leaf L of \mathcal{L} has finite genus at most k .*

Proof. First suppose that $L = L_p$ for some $p \in \mathcal{S}$. In this case \overline{L}_p is a plane and so L has genus zero, which implies that the claim holds for L .

Next consider the case $\overline{L} \cap \mathcal{S} = \emptyset$. In this case \overline{L} is a minimal lamination of \mathbb{R}^3 and the arguments in the proof of Theorem 1.3 imply that the claim holds for L .

Finally, consider the case that $p \in \overline{L} \cap \mathcal{S}$ and $L \neq L_p$. In this case, L lies in a halfspace component \mathcal{H} of $\mathbb{R}^3 - \overline{L}_p$. Suppose for the moment that $(\overline{L} \cap \mathcal{S}) - \overline{L}_p \neq \emptyset$ and let $q \in (\overline{L} \cap \mathcal{S}) - \overline{L}_p$. In this subcase, L is contained in the open slab \mathcal{T} of \mathbb{R}^3 with boundary planes \overline{L}_p and \overline{L}_q . Since L is connected, it must intersect every horizontal plane contained in \mathcal{T} and $\mathcal{T} \cap \mathcal{S} = \emptyset$.

We claim that L is properly embedded in \mathcal{T} . If not, then the closure of L in \mathcal{T} is a minimal lamination of \mathcal{T} with a limit leaf X , which is stable. By the same argument as in Claim 4.2, stability implies that X extends across the closed countable set $\overline{L} \cap \mathcal{S} \subset (\overline{L}_p \cup \overline{L}_q) \cap \mathcal{S}$ to a complete stable minimal surface in \mathbb{R}^3 . Hence, X is a horizontal plane in \mathcal{T} which is disjoint from L , which contradicts the discussion in the previous paragraph. Hence, in this subcase L is properly embedded in \mathcal{T} . A similar argument shows that if $(\overline{L} \cap \mathcal{S}) - \overline{L}_p = \emptyset$, then the leaf

L is properly embedded in the half space \mathcal{H} . Hence, in either case, L is properly embedded in an open simply-connected subset of \mathbb{R}^3 and so it separates this open set and has an open regular neighborhood in it. Now the arguments in the proof of Theorem 1.3 imply that L has finite genus at most k . \square

By using Claims 4.2 and 4.3, the next claim follows. Since the proof of Claim 4.4 is almost identical to the proof of the first statement in Claim 3.2 in [30], we omit it here.

Claim 4.4. *For any $t \in \mathbb{R}$, the intersection $\{x_3 = t\} \cap \mathcal{S}$ is nonempty.*

We now invoke the last hypothesis in the statement of Theorem 1.4:

$$\lim_{n \rightarrow \infty} I_{M_n}(\vec{0}) = \infty.$$

We remark that we have obtained Claim 4.4 without invoking this hypothesis. After replacing by a subsequence, for each $m \in \mathbb{N}$, there exists an increasing sequence $N(m) \in \mathbb{N}$ such that the injectivity radius function of $M_{N(m)}$ at $\vec{0}$ is greater than m/δ_1 , where δ_1 is the constant given in Theorem 2.8. Therefore, by the same theorem, for any $m \in \mathbb{N}$, the connected component $M(N(m))$ of $M_{N(m)} \cap \mathbb{B}(m)$ containing the origin is an $H_{N(m)}$ -disk with $\partial M(N(m)) \subset \partial \mathbb{B}(m)$. For simplicity of notation and after replacing by a further subsequence and relabeling, we will use M_n to denote the sequence $M_{N(m)}$ and so $M(n)$ will now denote $M(N(m))$. After replacing by a further sequence, we will assume that item B of Theorem 2.10 holds.

Let l denote the x_3 -axis. Since $\lim_{n \rightarrow \infty} |A_{M_n}|(\vec{0}) = \infty$, part (a) of item B of Theorem 2.10 shows that the sequence $M(n)$ converges away from l to a foliation \mathcal{L}' of $\mathbb{R}^3 - l$ by punctured horizontal planes. Part (b) of item B of Theorem 2.10 implies that given $R > 0$, if n is sufficiently large there exists a possibly disconnected compact subdomain $\mathcal{C}_n(R)$ of $M(n)$, with $[M(n) \cap \mathbb{B}(R/2)] \subset \mathcal{C}_n(R) \subset \mathbb{B}(R)$ and with $\partial \mathcal{C}_n(R) \subset \mathbb{B}(R) - \mathbb{B}(R/2)$, consisting of a disk $\mathcal{D}_n(R, 1)$ containing the origin $\vec{0}$ and possibly a second disk $\mathcal{D}_n(R, 2)$. Moreover, the diameter of each connected component of $\mathcal{C}_n(R)$ is bounded by $3R$ and $\mathcal{D}_n(R, i) \cap \mathbb{B}(R/n) \neq \emptyset$, for $i = 1, 2$. Hence, if $M_n \cap \mathbb{B}(R/2) = M(n) \cap \mathbb{B}(R/2)$ then the theorem follows. If that is not the case, then we proceed as follows.

Suppose, after choosing a subsequence, that for some $R > 0$, $M_n \cap \mathbb{B}(R/2)$ contains a component $\Delta_n(R)$ that is not contained in $\mathcal{C}_n(R)$. We first show that even in this case, the sequence $\{M_n\}_{n \in \mathbb{N}}$, and not solely $\{M(n)\}_{n \in \mathbb{N}}$, converges to the foliation \mathcal{L}' away from l . Since M_n has locally positive injectivity radius, the horizontal planar regions forming on $M(n)$ away from l imply that the sequence $\{M_n\}_{n \in \mathbb{N}}$ has locally bounded norm of the second fundamental form in

$\mathbb{R}^3 - l$; these curvature estimates arise from the intrinsic curvature estimates in Corollary 2.5. By the embeddedness of M_n , M_n must converge to \mathcal{L}' away from l as n goes to infinity and l is again a line and nearby it the sequence M_n has arbitrary large norm of the second fundamental form. This discussion proves that $\mathcal{S} = l$ and $\mathcal{L} = \mathcal{L}'$ regardless of whether or not $M_n \cap \mathbb{B}(R/2) = M(n) \cap \mathbb{B}(R/2)$. Moreover, using these curvature estimates and the double spiral staircase structure of $\mathcal{D}_n(R, 1)$, it is straightforward to prove that $\Delta_n(R)$ contains points y_n converging to $\vec{0}$.

After choosing a subsequence, for some R fixed and for every $n \in \mathbb{N}$, $M_n \cap \mathbb{B}(R/2) \neq M(n) \cap \mathbb{B}(R/2)$. In this remaining case let y_n be chosen as in the previous paragraph.

Assume that $\lim_n I_{M_n}(y_n) = \infty$; in fact, we will prove this in Claim 4.5. Arguing similarly to the previous discussion, after replacing by a subsequence, we may assume that $I_{M_n}(y_n) \geq n/\delta_1$, where δ_1 is the constant given in Theorem 2.8, and that $\partial M(n) \subset \partial \mathbb{B}(R_n)$, with $R_n > 2n$. By Theorem 2.8 the connected component $M'(n)$ of $M_n \cap \mathbb{B}(y_n, n)$ containing y_n is an H_n -disk with $\partial M'(n) \subset \partial \mathbb{B}(y_n, n)$. Item B of Theorem 2.10 implies that for n sufficiently large, there exists a possibly disconnected compact subdomain $\mathcal{C}'_n(R) \subset M'(n)$ with $\mathcal{C}'_n(R) \subset \mathbb{B}(y_n, R)$ and with $\partial \mathcal{C}'_n(R) \subset [\mathbb{B}(y_n, R) - \mathbb{B}(y_n, R/2)]$ consisting of a disk $\mathcal{D}'_n(R, 1)$ containing y_n and possibly a second disk $\mathcal{D}'_n(R, 2)$, where each disk has intrinsic diameter bounded by $3R$ and $\mathcal{D}'_n(R, i) \cap \mathbb{B}(y_n, R/n) \neq \emptyset$, for $i = 1, 2$.

Since $\lim_{n \rightarrow \infty} y_n = \vec{0}$ and $R_n > 2n$, $M(n)$ and $M'(n)$ are disks satisfying the following properties:

- $M(n) \subset \mathbb{B}(R_n)$ and $\partial M(n) \subset \partial \mathbb{B}(R_n)$;
- $M'(n) \subset \mathbb{B}(y_n, n) \subset \mathbb{B}(R_n)$ and $\partial M'(n) \subset \partial \mathbb{B}(y_n, n)$;
- $y_n \notin M(n)$ and $y_n \in M'(n)$.

Then elementary separation properties give that $M(n) \cap M'(n) = \emptyset$. In particular, $\mathcal{C}_n(R) \cap \mathcal{C}'_n(R) = \emptyset$. Thus, to finish the proof assuming $\lim_{n \rightarrow \infty} I_{M_n}(y_n) = \infty$, it suffices to show that $M_n \cap \mathbb{B}(R/2) \subset \mathcal{D}_n(R, 1) \cup \mathcal{D}'_n(R, 1)$.

Suppose that either $\mathcal{D}_n(R, 2)$ or $\mathcal{D}'_n(R, 2)$ existed. Applying Corollary 2.6 would give that $\vec{0}$ cannot be a singular point. Therefore, $\mathcal{D}_n(R, 2)$ and $\mathcal{D}'_n(R, 2)$ do not exist. On the other hand, if $M_n \cap \mathbb{B}(R/2) \neq [M(n) \cup M'(n)] \cap \mathbb{B}(R/2)$, then by repeating the arguments used so far, there would exist a sequence of point x_n with $\lim_{n \rightarrow \infty} x_n = \vec{0}$ and a third sequence of disks $\mathcal{D}''_n(R)$ disjoint from $\mathcal{D}_n(R, 1) \cup \mathcal{D}'_n(R, 1)$, with $x_n \in \mathcal{D}''_n(R)$ and $\partial \mathcal{D}''_n(R) \subset [\mathbb{B}(x_n, R) - \mathbb{B}(x_n, R/2)]$. Again, one would obtain a contradiction by applying Corollary 2.6. Therefore $M_n \cap \mathbb{B}(R/2) = [M(n) \cup M'(n)] \cap \mathbb{B}(R/2)$ and so, to complete the proof of Theorem 1.4, it remains to prove the claim below.

Claim 4.5. $\lim_{n \rightarrow \infty} I_{M_n}(y_n) = \infty$.

Proof. Arguing by contradiction, suppose that after replacing by a subsequence $\lim_{n \rightarrow \infty} [I_{M_n}(y_n) = T_n] = T \in (0, \infty)$. Since $\lim_{n \rightarrow \infty} H_n = 0$, then, for n large, the Gauss equation implies that the $\limsup K_{M_n}$ of the Gaussian curvature functions of the surfaces M_n is non-positive. Classical results on Jacobi fields along geodesics in such surfaces imply that for n large the exponential map of M_n at y_n on the closed disk in $T_{y_n} M_n$ of radius T_n is a local diffeomorphism that is injective on the interior of the disk but it is not injective along its boundary circle of radius T_n . Hence, there exists a sequence of simple closed geodesic loops $\alpha_n \subset M_n$ based at y_n and of lengths $2T_n$ converging to $2T$ that are smooth everywhere except possibly at y_n . Since the sequence $\{M_n\}_{n \in \mathbb{N}}$ has locally positive injectivity radius in \mathbb{R}^3 , there exists an $\varepsilon \in (0, T)$ such that for n large,

$$I_{M_n}|_{M_n \cap \mathbb{B}(\vec{0}, 5T)} \geq \varepsilon.$$

Therefore, if the intrinsic distance between two points x and y in $M_n \cap \mathbb{B}(\vec{0}, 5T)$ is less than ε , then there exists a unique length minimizing geodesic in $B_{M_n}(x, \varepsilon)$ connecting them.

Since the sequence of surfaces is converging to flat planes away from l and $\lim_{n \rightarrow \infty} y_n = \vec{0}$, if for some divergent sequence of integers n , there were points $p_n \in \alpha_n$ that lie outside of some fixed sized cylindrical neighborhood of l and converge to a point p , then a subsequence of the geodesics α_n would converge to a set containing an infinite geodesic ray starting at p in the horizontal plane containing p . This follows because the convergence is smooth away from l . If there were a sequence of points $p_n \in \alpha_n$ converging to a point p not in l then a neighborhood U_n of p_n would converge smoothly to a horizontal flat disk $D(p)$ centered at p . Since α_n is a geodesic, $\alpha_n \cap U_n$ would converge to a diameter d of $D(p)$ and there would be a point $q \in D(p)$ which is the limit of points $q_n \in \alpha_n$ and that is further away from l than p . The convergence is smooth nearby q . Therefore, applying the previous argument gives that the limit set of convergence of α_n can be extended at q in the direction \vec{pq} . Iterating this argument would give that the limit set of α_n contains an infinite geodesic ray starting at p in the horizontal plane containing p . This would give a contradiction because the loops have length less than $3T$. Therefore after replacing by a subsequence, the α_n must converge to a vertical segment σ containing the origin and of length less than or equal to T .

Note that by Theorem 2.8, for n large, α_n cannot be contained in $\mathbb{B}(y_n, \delta_1 \varepsilon)$, otherwise it would be contained in $B_{M_n}(y_n, \varepsilon/2)$ and, by the properties of ε , $B_{M_n}(y_n, \varepsilon/2)$ cannot contain a geodesic loop such as α_n . Therefore, if we let p_1 and p_2 be the endpoints of the line segment σ , without loss of generality, we can assume that $p_1 \neq \vec{0}$ and that $x_3(p_1) \in [\delta_1 \varepsilon, T]$. Let q_n be points of α_n with a

largest x_3 -coordinate, and so $\lim_{n \rightarrow \infty} q_n = p_1$. By arguments similar to the ones used in the previous paragraphs of this proof, for any $r < \varepsilon/2$, $\alpha_n \cap \mathbb{B}(q_n, \delta_1 r)$ contains an arc component β_n with $\partial\beta_n = z_n(1) \cup z_n(2) \subset \partial\mathbb{B}(q_n, \delta_1 r)$ satisfying the following properties for n large:

1. $q_n \in \beta_n$;
2. $\lim_{n \rightarrow \infty} |z_n(1) - z_n(2)| = 0$;
3. $z_n(2) \in B_{M_n}(z_n(1), r)$
4. $\text{dist}_{M_n}(z_n(1), z_n(2)) \geq \delta_1 r$.

Let $r := \min\{\varepsilon/a, \varepsilon/2\}$, where a is the constant given in Theorem 2.9. Then applying Theorem 2.9 with $\vec{0}$ replaced by $z_n(1)$ and $R = r$, we have that if $\sup_{B_\Sigma(z_n(1), r_0(n))} |A_{M_n}| > \frac{1}{r_0(n)}$ where $r > r_0(n)$, then

$$\frac{1}{3} \text{dist}_{M_n}(z_n(1), z_n(2)) < |z_n(1) - z_n(2)| + r_0(n).$$

Since as n goes to infinity, there are points arbitrarily intrinsically close to $z_n(1)$ and with arbitrarily large norm of the second fundamental form, we can assume that $r_0(n) < \frac{\delta_1 \varepsilon}{6a}$. Combining this, $\text{dist}_{M_n}(z_n(1), z_n(2)) \geq \frac{\delta_1 \varepsilon}{a}$ and the previous inequality, we have obtained that

$$\frac{\delta_1 \varepsilon}{6a} < |z_n(1) - z_n(2)|.$$

Since the right hand-side of this inequality is going to zero as n goes to infinity, while the left hand-side is fixed, bounded away from zero, independently on n , we have obtained a contradiction, which finishes the proof that $\lim_{n \rightarrow \infty} I_{M_n}(y_n) = \infty$. \square

Now that item 2 of Theorem 1.4 is proved, we can apply Theorem 2.10 and Remark 2.11 to obtain the double spiral staircase description in item 3 for the each of the 1 or 2 components of \mathcal{C}_n .

This final observation completes the proof of Theorem 1.4.

5 The proof of Theorem 1.5.

In this section we will prove Theorem 1.5. Suppose that $\{M_n\}_{n \in \mathbb{N}}$ is a sequence of compact H_n -surfaces in \mathbb{R}^3 with finite genus at most k , $\vec{0} \in M_n$, M_n contains

no spherical components, $\partial M_n \subset [\mathbb{R}^3 - \mathbb{B}(n)]$, the sequence has locally positive injectivity radius in \mathbb{R}^3 and

$$\lim_{n \rightarrow \infty} |A_{M_n}|(\vec{0}) = \infty \text{ and } \lim_{n \rightarrow \infty} I_{M_n}(\vec{0}) = C,$$

for some $C > 0$.

After replacing by a subsequence, Claim 4.4 implies that the surfaces M_n converge C^α , for any $\alpha \in (0, 1)$, to a minimal lamination \mathcal{L} outside of a closed set \mathcal{S} with $\vec{0} \in \mathcal{S}$ and $x_3(\mathcal{S}) = \mathbb{R}$, where each leaf of \mathcal{L} is a horizontal plane punctured in a discrete set of points in \mathcal{S} .

The Colding-Minicozzi picture of \mathcal{L} around each point of \mathcal{S} together with the curvature estimates in Theorem 2.4 and the fact that the foliation \mathcal{L} is a foliation of $\mathbb{R}^3 - \mathcal{S}$ by punctured horizontal planes imply that if \mathcal{S}_0 is a connected component of \mathcal{S} , then $x_3(\mathcal{S}_0) = \mathbb{R}$ and \mathcal{S}_0 is a Lipschitz graph over the x_3 -axis. Moreover, given $p \in \mathcal{S} \cap \mathbb{B}(R)$, there exists $\delta := \delta(R) > 0$ such that for n large, the intersection $M_n \cap \mathbb{B}(p, \delta)$ consists of one or two disks. This is a consequence of Corollary 2.6, Theorem 2.8 and the fact that for n large, the injectivity radius function of M_n is bounded away from zero on any fixed compact set of \mathbb{R}^3 . By this observation and arguing like in the proof of Theorem 1.1 in [30], one obtains the following result.

Claim 5.1. *The set \mathcal{S} satisfies the following properties:*

- *The set \mathcal{S} is a discrete collection of vertical lines, one of which is the x_3 -axis.*
- *Given $R > 0$, for n sufficiently large the intersection of each line segment in $\mathcal{S} \cap \mathbb{B}(R)$ is the C^1 limit with multiplicity at most two of analytic curves in M_n which are pre-images of the equator via the Gauss map.*
- *Let l be a line in \mathcal{S} . Given $p \in l$ and $R > 0$ such that $\mathbb{B}(p, R) \cap \mathcal{S} \subset l$ then, for n large, the collection \mathcal{C}_n of components of $M_n \cap \mathbb{B}(p, \frac{R}{2})$ such that $\mathcal{C}_n \cap \mathbb{B}(p, \frac{R}{4}) \neq \emptyset$ consists of at most two disjoint disks. Furthermore, each of the 1 or 2 disk components of \mathcal{C}_n is contained in a disk in $M_n \cap \mathbb{B}(p, R)$ with boundary curve in $\mathbb{B}(p, R) - \mathbb{B}(p, R/2)$, where these disks have the structure of double spiral staircases, see Remark 2.11, with central columns that are graphs with small C^1 -norms over an arc in $l \cap \mathbb{B}(R)$, and $\mathcal{C}_n \cap \mathbb{B}(p, \frac{R}{4})$ is contained in the union of these subdisks.*

Let l be a line in \mathcal{S} . We now need to attach two labels to l . The first one is the following: if l is the C^1 limit with multiplicity one, respectively two, of analytic curves which are pre-images of the equator via the Gauss map, we say that l has *multiplicity one*, respectively *two*. The second label is the following: let $\mathcal{C}_l(R)$ be the vertical solid cylinder of radius R with axis l . For a given line l in \mathcal{S} , fix $R_l > 0$

such that $C_l(2R_l) \cap \mathcal{S} = l$. Then, for n large, $\partial C_l(R_l/2) \cap M_n$ contains either two or four highly winding spirals nearby the (x_1, x_2) -plane. Since M_n is embedded, these spirals are all right-handed or left-handed for a given n . After passing to a subsequence and using a diagonal argument gives that for a given l in \mathcal{S} and n large, the spirals have the same “handedness.” We say that l is *right-handed* if such spirals are right-handed and that l is *left-handed* otherwise.

In the next claims we prove that \mathcal{S} consists of exactly two vertical lines.

Claim 5.2. *Let l_1 and l_2 be two distinct components of \mathcal{S} . Then, if l_1 is right-handed (left-handed), l_2 must be left-handed (right-handed). In particular, \mathcal{S} consists of at most two lines.*

Proof. Arguing by contradiction, suppose that l_1 and l_2 are two distinct components of \mathcal{S} having the same handedness. We will obtain a contradiction by proving that as n goes to infinity, the number of pairwise disjoint pairs of loops in M_n such that each pair intersects transversely at one point is greater than the fixed genus bound k for the surfaces M_n . By using standard topological arguments, the existence of such loops implies that the genus M_n is greater than k .

Without loss of generality, suppose that l_1 and l_2 are both left-handed. Let $p_i := l_i \cap \{x_3 = 0\}$, $i = 1, 2$ and let $\overline{p_1 p_2}$ denote the line segment connecting them. For simplicity, first assume that $\overline{p_1 p_2} \cap [\mathcal{S} - [l_1 \cup l_2]] = \emptyset$. Then, as n goes to infinity, the segment $\overline{p_1 p_2} - [C_{l_1}(R_{l_1}/4) \cup C_{l_2}(R_{l_2}/4)]$ lifts near the (x_1, x_2) -plane to an increasing number of arcs γ_i in $M_n - [C_{l_1}(R_{l_1}/4) \cup C_{l_2}(R_{l_2}/4)]$. In fact, an ε -neighborhood Γ of $\overline{p_1 p_2}$ in the (x_1, x_2) -plane lifts to an increasing number of strips Γ_i in $M_n - [C_{l_1}(R_{l_1}/4 - \varepsilon) \cup C_{l_2}(R_{l_2}/4 - \varepsilon)]$. Because M_n is embedded, the strips Γ_i can be ordered by their relative heights. Moreover, the arcs of the spiralling curves $M_n \cap \partial C_{l_1}(R_{l_1}/2)$ given by $\Gamma_i \cap \partial C_{l_1}(R_{l_1}/2)$ can be connected, via arcs in Γ_i , to the arcs in the spiralling curves $M_n \cap \partial C_{l_2}(R_{l_2}/2)$ given by $\Gamma_i \cap \partial C_{l_2}(R_{l_2}/2)$. There are three possibilities to consider.

1. The lines l_1 and l_2 have both multiplicity one.
2. One line has multiplicity one and the other one has multiplicity two.
3. Both lines have multiplicity two.

The construction of the collection of pairwise disjoint pairs of loops when the lines l_1 and l_2 have both multiplicity one is illustrated in Figure 3. The construction of the collection of pairwise disjoint pairs of loops in case two is illustrated in Figure 4. The construction in the third and last case is also straightforward and it is left to the reader.

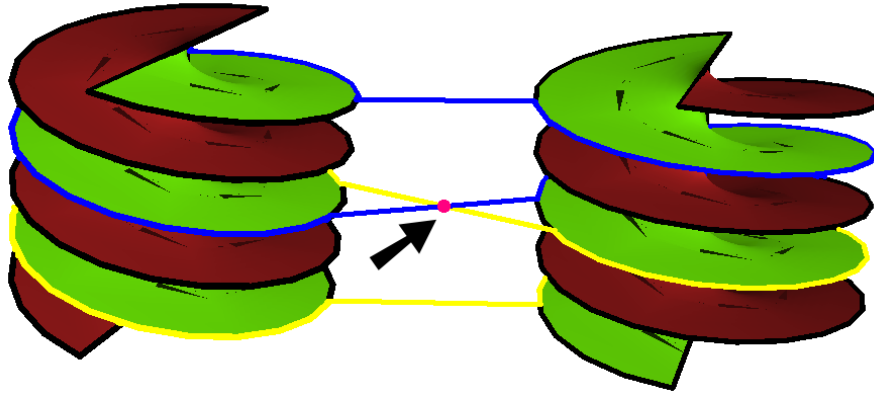


Figure 3: The blue curve and the yellow curve intersect exactly at one point.

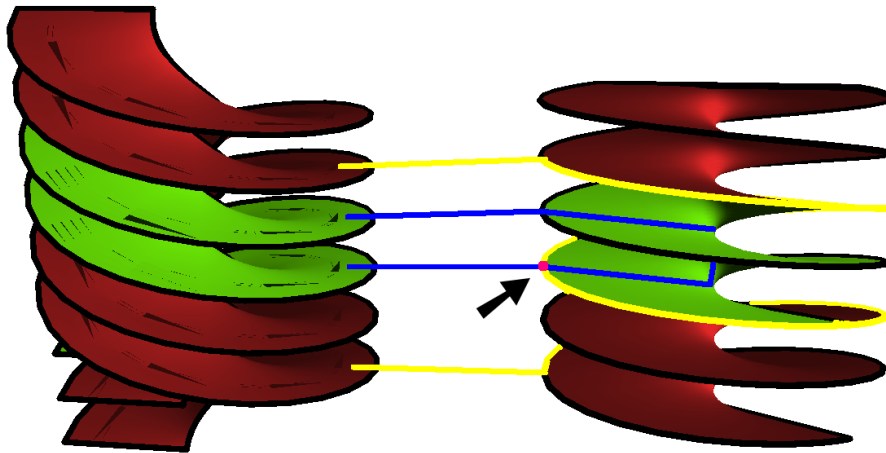


Figure 4: The right side of the picture is connected. On the left side of the picture, the red set is part of a connected set \mathcal{H}_1 and the green set is part of a connected set \mathcal{H}_2 . The sets \mathcal{H}_1 and \mathcal{H}_2 are disjoint. Therefore, the end points of the blue arc and of the yellow arc can be connected so that the resulting closed curves intersect in exactly one point, as shown in the picture.

If $\overline{p_1 p_2} \cap [\mathcal{S} - [l_1 \cup l_2]] \neq \emptyset$, the proof can be easily modified by replacing $\overline{p_1 p_2}$ by a smooth embedded arc in the (x_1, x_2) -plane that is a small normal graph over $\overline{p_1 p_2}$ and only intersects the singular set at its end points p_1, p_2 . \square

The next claim finally shows that \mathcal{S} consists of exactly two lines. Note that by the previous claim, if there are two lines in \mathcal{S} , then one of these two lines must be right-handed and the other one must be left-handed. Recall that

$$\lim_{n \rightarrow \infty} I_{M_n}(\vec{0}) = C.$$

Claim 5.3. *The set \mathcal{S} consists of exactly two vertical lines one of which is the x_3 -axis.*

Proof. We have already shown that there are at most two vertical lines in \mathcal{S} and that the x_3 -axis is in \mathcal{S} . Since $\lim_{n \rightarrow \infty} H_n = 0$, then, for n large, the Gauss equation implies that the $\limsup K_{M_n}$ of the Gaussian curvature functions of the surfaces M_n is non-positive. Since $\lim_{n \rightarrow \infty} [I_{M_n}(\vec{0}) = C_n] = C$, classical results on Jacobi fields along geodesics in such surfaces imply that for n large the exponential map of M_n at the origin on the closed disk in $T_{\vec{0}} M_n$ of radius C_n is a local diffeomorphism that is injective on the open disk but not injective along its boundary circle of radius C_n . Hence, there exists a sequence of simple closed geodesic loops $\alpha_n \subset M_n$ based at the origin and of lengths $2C_n$ converging to $2C$ that are smooth everywhere except possibly at the origin. Arguing by contradiction, suppose that \mathcal{S} is the x_3 -axis. The arguments to rule out this picture are exactly the same ones used in the proof of Claim 4.5 by taking $T = C$. This implies that the number of lines in \mathcal{S} must be two. \square

From now on, l_1 will denote the x_3 -axis, l_2 will denote the other component in \mathcal{S} and $p_2 = l_2 \cap \{x_3 = 0\}$.

Claim 5.4. *If l_1 has multiplicity one, respectively two, then so does l_2 .*

Proof. By Claim 5.2, one vertical line must be right-handed and the other must be left-handed. Suppose that one line has multiplicity one and the other has multiplicity two. Like in the proof of Claim 5.2, we will obtain a contradiction by proving that as n goes to infinity, the number of pairwise disjoint pairs of loops such that each pair intersects transversely at one point is greater than the fixed genus bound k for the surfaces M_n . The construction of such pairs of loops is illustrated in Figure 5. \square

Assume now that l_1 has multiplicity two. Let $d > 0$ denote the distance between l_1 and l_2 . Recall that by Claims 5.1 and 5.4, if l_1 has multiplicity two so does

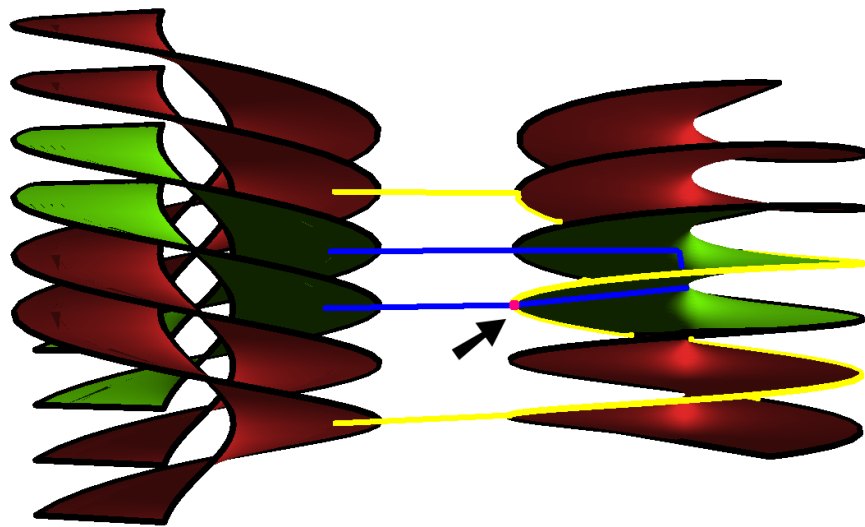


Figure 5: The right side of the picture is connected. On the left side of the picture, the red set is part of a connected set \mathcal{H}_1 and the green set is part of a connected set \mathcal{H}_2 . The sets \mathcal{H}_1 and \mathcal{H}_2 are disjoint. Therefore, the end points of the blue arc and of the yellow arc can be connected so that the resulting closed curves intersect in exactly one point, as shown in the picture. One of the differences with Figure 4 is in the handedness of the lines.

l_2 and that, for n large, the subset $\mathcal{D}_1(n)$ of $M_n \cap \mathbb{B}(\frac{d}{2})$ such that $\mathcal{D}_1(n) \cap \mathbb{B}(\frac{d}{4}) \neq \emptyset$ consists of two disks $A_1(n)$ and $B_1(n)$ and the subset $\mathcal{D}_2(n)$ of $M_n \cap \mathbb{B}(p_2, \frac{d}{2})$ such that $\mathcal{D}_2(n) \cap \mathbb{B}(p_2, \frac{d}{4}) \neq \emptyset$ consists of two disks $A_2(n)$ and $B_2(n)$.

Claim 5.5. *Let $R > d$ and suppose that l_1 has multiplicity two. Then, for n large, after possibly relabeling the disks, the following holds. The components Δ_n of $M_n \cap \mathbb{B}(R)$ such that $\Delta_n \cap \mathbb{B}(\frac{d}{4}) \neq \emptyset$ consist of two distinct planar domains $\Delta_1(n)$ and $\Delta_2(n)$ such that $A_1(n) \cup A_2(n) \subset \Delta_1(n)$ and $B_1(n) \cup B_2(n) \subset \Delta_2(n)$.*

Proof. Let Π denote the vertical plane perpendicular to the segment $\vec{0}p_2$ connecting $\vec{0}$ and p_2 and containing its midpoint. As n goes to infinity, $\Pi \cap \Delta_n$ consists of an increasing collection of arcs $S(n)$ that are becoming horizontal arcs. Since Δ_n is embedded, these planar arcs can be ordered by their relative heights over the midpoint of $\vec{0}p_2$. Note that $\Delta_n - \Pi$ contains four disconnected components $\Omega_{ij}(n)$, $i, j = 1, 2$, and the following holds for $i = 1, 2$: $A_i(n) \subset \Omega_{i1}(n)$ and $B_i(n) \subset \Omega_{i2}(n)$. For $i = 1, 2$ let $\alpha_i(n)$ denote the arcs in $S(n)$ that are contained in the boundary of $\Omega_{i1}(n)$ and let $\beta_i(n)$ denote the arcs in $S(n)$ that are contained in the boundary of $\Omega_{i2}(n)$. Recall that $A_1(n)$ and $B_1(n)$, respectively $A_2(n)$ and $B_2(n)$, separate $\mathbb{B}(\frac{d}{4})$, respectively $\mathbb{B}(p_2, \frac{d}{4})$, into three components and the mean curvature vector of M_n points outside of the component $W_1(n)$, respectively $W_2(n)$, with boundary $A_1(n) \cup B_1(n)$, respectively $A_2(n) \cup B_2(n)$. This is because otherwise applying Corollary 4.9 in [30] would give curvature estimates in a neighborhood of $\vec{0}$, respectively p_2 , and this would contradict the fact that $\vec{0}$, respectively p_2 , is in \mathcal{S} . Using this observation and the previous discussion gives that either $\alpha_1(n) = \alpha_2(n)$ and $\beta_1(n) = \beta_2(n)$, or $\alpha_1(n) = \beta_2(n)$ and $\beta_1(n) = \alpha_2(n)$. After possibly relabeling, either case implies that Δ_n is disconnected.

It remains to prove that each connected component of Δ_n has genus zero. This follows from the “almost periodicity” of the previous description. If there were a pair of loops intersecting at exactly one point then, as n goes to infinity, there would be an increasing number of such pairs in M_n , contradicting the fact that the genus of M_n is bounded from above by k . \square

Remark 5.6. If instead l_1 has multiplicity one then, by the same arguments, the components Δ_n of $M_n \cap \mathbb{B}(R)$ such that $\Delta_n \cap \mathbb{B}(\frac{d}{4}) \neq \emptyset$ consist of a unique planar domain for n large. Moreover, it is easy to see that if l_1 has multiplicity two and $\Delta_2(n)$ denotes the connected component of Δ_n that intersect $\mathbb{B}(\frac{d}{4})$ and does not contain the origin then, after possibly reindexing the subsequence, $\Delta_2(n) \cap \mathbb{B}(\frac{R}{n}) \neq \emptyset$.

For the time being, let us assume that the distance between l_1 and l_2 is C . We now deal with the construction of closed curves with non-zero flux. Note that

this construction is analogous to the one described in Figures 4, 5 and 6 in [23], which in turn was a modification of a related argument in [22]; the closed curves constructed by the methods in [22, 23] are called *connection loops*.

Recall that if γ is a 1-cycle in an H -surface M , then the *flux* of γ is

$$F(\gamma) = \int_{\gamma} (H\gamma + \xi) \times \dot{\gamma},$$

where ξ is the unit normal to M along γ . The flux of a 1-cycle in M is a homological invariant.

Given $\varepsilon > 0$ sufficiently small, as n goes to infinity, the line segment $\vec{0}p_2 - [\mathbb{B}(\varepsilon) \cup \mathbb{B}(p_2, \varepsilon)]$ lifts to an increasing number of arcs $\gamma_i(n, \varepsilon)$ in $M_n - [\mathbb{B}(\varepsilon) \cup \mathbb{B}(p_2, \varepsilon)]$ that, as n goes to infinity, converge C^1 to the line segment $\vec{0}p_2 - [\vec{0}p_2 \cap [\mathbb{B}(\varepsilon) \cup \mathbb{B}(p_2, \varepsilon)]]$. Because M_n is embedded, the lifts $\gamma_i(n, \varepsilon)$ can be ordered by their relative heights and the signs of the inner product between the unit normal vector to M_n along $\gamma_i(n, \varepsilon)$ and $(0, 0, 1)$ are alternating.

Let ε_n be a sequence of positive numbers with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ such there exists a sequence of two consecutive lifts $\gamma_1(n, \varepsilon_n)$ and $\gamma_2(n, \varepsilon_n)$ of $\vec{0}p_2 - [\mathbb{B}(\varepsilon_n) \cup \mathbb{B}(p_2, \varepsilon_n)]$ and the following holds: the end points of such lifts are contained in $\mathbb{B}(2\varepsilon_n)$ and $\mathbb{B}(p_2, 2\varepsilon_n)$ and the lifts converge to the line segment $\vec{0}p_2$ away from $\vec{0}$ and p_2 as n goes to infinity. Let $\alpha_1(n, \varepsilon_n)$ be an arc in $\mathbb{B}(2\varepsilon_n) \cap M_n$ connecting the endpoints of $\gamma_1(n, \varepsilon_n)$ and $\gamma_2(n, \varepsilon_n)$ in $\mathbb{B}(\varepsilon_n)$ and let $\alpha_2(n, \varepsilon_n)$ be an arc in $\mathbb{B}(p_2, 2\varepsilon_n) \cap M_n$ connecting the endpoints of $\gamma_1(n, \varepsilon_n)$ and $\gamma_2(n, \varepsilon_n)$ in $\mathbb{B}(p_2, 2\varepsilon_n)$ such that the loop

$$\gamma_1(n, \varepsilon_n) \cup \alpha_1(n, \varepsilon_n) \cup \gamma_2(n, \varepsilon_n) \cup \alpha_2(n, \varepsilon_n)$$

is smooth; note that since the sequence $\{M_n\}_{n \in \mathbb{N}}$ has locally positive injectivity radius in \mathbb{R}^3 , by using Theorem 2.8, as n goes to infinity, the sum of the lengths of the arcs $\alpha_1(n, \varepsilon_n)$ and $\alpha_2(n, \varepsilon_n)$, can be assumed to approach zero as well. Let Γ_n be a unit speed parametrization of such a loop and, for $p \in \Gamma_n$ let $N_n(p)$ denote the normal to M_n at p .

Recall that as n goes to infinity, the mean curvature of M_n is going to zero therefore, since the length of Γ_n is bounded from above independently of n , the term in the flux formula involving the mean curvature is going to zero. In other words,

$$F(\Gamma_n) = \int_{\Gamma_{(n, \varepsilon)}} N_n(p) \times \dot{\Gamma}_n(p) + f(n), \quad \text{where } \lim_{n \rightarrow \infty} f(n) = 0.$$

As n goes to infinity, for any $p \in \gamma_i(n, \varepsilon_n)$ the vectors $N_n(p) \times \dot{\Gamma}_n(p)$ are converging to the same unit vector perpendicular to $\vec{0}p_2$, the lengths of $\alpha_1(n, \varepsilon_n) \cup$

$\alpha_2(n, \varepsilon_n)$ are going to zero, and the lengths of $\gamma_1(n, \varepsilon_n) \cup \gamma_2(n, \varepsilon_n)$ are converging to $2C$. Therefore, after possibly changing their orientation, the curves Γ_n converge to the line segment $\vec{0}p_2$, have lengths converging to $2C$ and fluxes converging to $(0, 2C, 0)$. This construction of curves with non-zero flux finishes the proof of item 3 of Theorem 1.5, assuming that the distance between the lines l_1 and l_2 is C .

We can now prove that the distance d between the lines l_1 and l_2 is C . Arguing by contradiction, suppose that $d < C$ or $d > C$. If $d < C$ then, by the previous arguments, there exists a sequence of loops Γ_n containing the origin with the norms of their fluxes bounded from below by d . Since the flux of a 1-cycle is a homological invariant, this implies that such curves are homologically non-trivial. Moreover the lengths of Γ_n are converging to $2d < 2C$. Therefore, there exists $\varepsilon > 0$ such that for n sufficiently large, $\Gamma_n \subset B_{M_n}(\vec{0}, C - \varepsilon)$. However, since $\lim_{n \rightarrow \infty} [I_{M_n}(\vec{0}) = C_n] = C$, for n sufficiently large $B_{M_n}(\vec{0}, C - \varepsilon)$ is a disk. This implies that for n sufficiently large, Γ_n is homologically trivial which is a contradiction.

Suppose $d > C$. Since $\lim_{n \rightarrow \infty} [I_{M_n}(\vec{0}) = C_n] = C \in (0, \infty)$, there exists a sequence of simple closed geodesic loops $\alpha_n \subset M_n$ based at $\vec{0}$ and of lengths $2C_n$ converging to $2C$ that are smooth everywhere except possibly at $\vec{0}$; see the proof of Claim 4.5. In fact, arguing exactly as in the proof of Claim 4.5 gives that the limit set of α_n must contain a point in $\mathcal{S} - l_1$. Note that $\alpha_n \subset \mathbb{B}(C_n)$. Since $d > C$, there exists $\varepsilon > 0$ such that, for n sufficiently large, $\alpha_n \subset \mathbb{B}(d - \varepsilon)$. In particular, for n sufficiently large, α_n is at distance at least ε from the line $l_2 = \mathcal{S} - l_1$ and thus the limit set of α_n does not contain a point in $\mathcal{S} - l_1$. This contradiction proves that the distance between l_1 and l_2 is equal to C .

Finally, given $R > C$ let $\Delta(n)$ be a connected component of $M_n \cap \mathbb{B}(R)$ that intersects $\mathbb{B}(\frac{R}{4})$. We want to show that for n sufficiently large, given two points in $\Delta(n)$, their distance in M_n is less than $3R$. Without loss of generality, let us assume that $\Delta(n)$ is the connected component containing the origin. Then, it suffices to show that given a point in $\Delta(n)$, its distance to the origin in M_n is less than $\frac{3}{2}R$. Arguing by contradiction, assume there exists $R > C$ and points $p(n) \in \Delta(n)$ at distance greater than or equal to $\frac{3}{2}R$ to the origin. After going to a subsequence, let $p = \lim_{n \rightarrow \infty} p(n)$. Recall that by Theorem 2.8 and Claim 5.5, since the sequence $\{M_n\}_{n \in \mathbb{N}}$ has locally positive injectivity radius in \mathbb{R}^3 , there exists $\varepsilon > 0$ such that for n sufficiently large, the intersection $\Delta(n) \cap \mathbb{B}(\varepsilon)$ is a disk that is contained in $B_{M_n}(R)$. Therefore, for n sufficiently large, $p_n \notin \mathbb{B}(\varepsilon)$ which implies that $p \notin \mathbb{B}(\frac{\varepsilon}{2})$.

Let γ be the horizontal line segment connecting p to a point q in the x_3 -axis and let α be the line segment in the x_3 -axis connecting q to the origin. If $\gamma \cap l_2 \neq \emptyset$, let z denote such point of intersection. Note that the length of $\gamma \cup \alpha$ is less than $\sqrt{2}R < \frac{3}{2}R$. By the arguments used in the proof of this theorem, it is clear that

there exists a sequence of curves $\gamma(n)$ in M_n connecting the origin to the point $p(n)$ that converges to $\gamma \cup \alpha$ away from the points q and z . Moreover, the lengths of this curve converge to the length of $\gamma \cup \alpha$. Therefore, for n sufficiently large, the length of $\gamma(n)$ is less than $\frac{3}{2}R$ and so the distance from $p(n)$ to the origin in M_n is less than $\frac{3}{2}R$. This contradiction proves that for n sufficiently large, given two points in $\Delta(n)$, their distance in M_n is less than $3R$.

The geometric description given in item 2 of Theorem 1.5 follows easily from the arguments used in its proof. This finishes the proof of the theorem.

Remark 5.7. Suppose that for some $\varepsilon > 0$, $\{M_n\}_{n \in \mathbb{N}}$ is a sequence of compact H_n -surfaces in \mathbb{R}^3 with finite genus at most k , $\vec{0} \in M_n$, $d_{M_n}(\vec{0}, \partial M(n)) \rightarrow \infty$ and that $I_{M(n)}(x) \geq \varepsilon$ for any $x \in M(n)$ with $d_{M(n)}(x, \partial M(n)) > 1$. Then Corollary 3.2 in [28] shows that after replacing by a subsequence, the components M_n of $M(n) \cap \mathbb{B}(n)$ containing the origin satisfy the conditions of one of the Theorem 1.3, 1.4 or 1.5.

Definition 5.8. A point of *almost-minimal injectivity radius* of a compact surface M surface with boundary is a point $p \in M$ where the function $\frac{d_M(p, \partial M)}{I_M(p)}$ has its maximal value.

As a consequence of Remark 5.7, we have the following proposition that is related to Theorem 1.1 in [19], which was proved under the hypothesis that $H = 0$.

Proposition 5.9. *Let $M(n)$ be a sequence of compact H_n -surfaces with boundary embedded in \mathbb{R}^3 with finite genus at most k together with points $p_n \in M_n$ satisfying*

$$\lim_{n \rightarrow \infty} \frac{d_{M(n)}(p_n, \partial M(n))}{I_{M(n)}(p_n)} = \infty.$$

Given points $q_n \in M(n)$ of almost-minimal injectivity radius, there exist positive numbers R_n , $\lim_{n \rightarrow \infty} R_n = \infty$, satisfying:

1. *The component M_n of $[\frac{1}{I_{M(n)}(q_n)}(M(n) - q_n)] \cap \mathbb{B}(R_n)$ containing $\vec{0}$ has its boundary in $\partial \mathbb{B}(R_n)$ and genus at most k .*
2. *The sequence $\{M_n\}_{n \in \mathbb{N}}$ has uniformly positive injectivity radius in \mathbb{R}^3 and $I_{M_n}(\vec{0}) = 1$.*

Then after choosing a subsequence and then translating the surfaces M_n by vectors of length at most 1, the sequence $\{M_n\}_{n \in \mathbb{N}}$ satisfies the hypotheses of Theorems 1.5 with $C = 1$ or the sequence $\{M_n\}_{n \in \mathbb{N}}$ satisfies the hypotheses of Theorem 1.3.

Proof. After choosing a subsequence suppose that

$$\frac{d_{M(n)}(p_n, \partial M(n))}{I_{M(n)}(p_n)} \geq n.$$

Let $q_n \in M_n$ be points of almost-minimal injectivity radius and let M'_n be the scaled and translated H'_n -surface

$$M'_n = [\frac{1}{I_{M(n)}(q_n)}[B_{M(n)}(q_n, n/2) - q_n]].$$

Observe that $\{M'_n\}_{n \in \mathbb{N}}$ is a sequence of compact H_n -surfaces in \mathbb{R}^3 with genus at most k , $\vec{0} \in M'_n$, $\lim_{n \rightarrow \infty} d_{M'_n}(\vec{0}, \partial M'_n) = \infty$ and that $I_{M'_n}(x) \geq 1/2$ for any $x \in M'_n$ with $d_{M'_n}(x, \partial M'_n) > 1$. Corollary 5.9 now follows immediately from Remark 5.7. \square

6 Appendix.

In this appendix we give the definition of a weak CMC lamination of a Riemannian three-manifold. Specializing to the case where all of the leaves have the same mean curvature $H \in \mathbb{R}$, one obtains the definition of a weak H -lamination, for which we give a few more explanations. A simple example of a weak 1-lamination \mathcal{L} of \mathbb{R}^3 that is not a 1-lamination is the union of two spheres of radius 1 that intersect at single point of tangency.

For further background material on these notions see Section 3 of [18], [25] or our previous papers [32, 33].

Definition 6.1. A (codimension-one) *weak CMC lamination* \mathcal{L} of a Riemannian three-manifold N is a collection $\{L_\alpha\}_{\alpha \in I}$ of (not necessarily injectively) immersed constant mean curvature surfaces called the *leaves* of \mathcal{L} , satisfying the following four properties.

1. $\bigcup_{\alpha \in I} L_\alpha$ is a closed subset of N . With an abuse of notation, we will also consider \mathcal{L} to be the closed set $\bigcup_{\alpha \in I} L_\alpha$.
2. The function $|A_{\mathcal{L}}|: \mathcal{L} \rightarrow [0, \infty)$ given by

$$|A_{\mathcal{L}}|(p) = \sup\{|A_L|(p) \mid L \text{ is a leaf of } \mathcal{L} \text{ with } p \in L\}. \quad (3)$$

is uniformly bounded on compact sets of N .

3. For every $p \in N$, there exists an $\varepsilon_p > 0$ such that if for some $\alpha \in I$, $q \in L_\alpha \cap B_N(p, \varepsilon_p)$, then q contains a disk neighborhood in L_α whose boundary is contained in $N - B_N(p, \varepsilon_p)$.

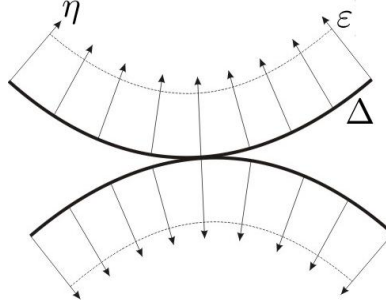


Figure 6: The leaves of a weak H -lamination with $H \neq 0$ can intersect each other or themselves, but only tangentially with opposite mean curvature vectors. Nevertheless, on the mean convex side of these locally intersecting leaves, there is a lamination structure.

4. If $p \in N$ is a point where either two leaves of \mathcal{L} intersect or a leaf of \mathcal{L} intersects itself, then each of these surfaces nearby p lies at one side of the other (this cannot happen if both of the intersecting leaves have the same signed mean curvature as graphs over their common tangent space at p , by the maximum principle).

Furthermore:

- If $N = \bigcup_{\alpha} L_{\alpha}$, then we call \mathcal{L} a *weak CMC foliation* of N .
- If the leaves of \mathcal{L} have the same constant mean curvature H , then we call \mathcal{L} a *weak H -lamination* of N (or H -foliation, if additionally $N = \bigcup_{\alpha} L_{\alpha}$).

The following proposition follows immediately from the definition of a weak H -lamination and the maximum principle for H -surfaces.

Proposition 6.2. *Any weak H -lamination \mathcal{L} of a Riemannian three-manifold N has a local H -lamination structure on the mean convex side of each leaf. More precisely, given a leaf L_{α} of \mathcal{L} and given a small disk $\Delta \subset L_{\alpha}$, there exists an $\varepsilon > 0$ such that if (q, t) denotes the normal coordinates for $\exp_q(t\eta_q)$ (here \exp is the exponential map of N and η is the unit normal vector field to L_{α} pointing to the mean convex side of L_{α}), then the exponential map \exp is an injective submersion in $U(\Delta, \varepsilon) := \{(q, t) \mid q \in \text{Int}(\Delta), t \in (-\varepsilon, \varepsilon)\}$, and the inverse image $\exp^{-1}(\mathcal{L}) \cap \{q \in \text{Int}(\Delta), t \in [0, \varepsilon)\}$ is an H -lamination of $U(\Delta, \varepsilon)$ in the pulled back metric, see Figure 6.*

Definition 6.3. A leaf L_α of a weak H -lamination \mathcal{L} is a *limit leaf* of \mathcal{L} if at some $p \in L_\alpha$, on its mean convex side near p , it is a limit leaf of the related local H -lamination given in Proposition 6.2.

- Remark 6.4.** 1. A weak H -lamination for $H = 0$ is a minimal lamination.
2. Every CMC lamination (resp. CMC foliation) of a Riemannian three-manifold is a weak CMC lamination (resp. weak CMC foliation).
3. Theorem 4.3 in [24] states that the 2-sided cover of a limit leaf of a weak H -lamination is stable. By Lemma 3.3 in [20] and the main theorem in [35], the only complete stable H -surfaces in \mathbb{R}^3 are planes. Hence, every leaf L of a weak H -lamination \mathcal{L} of \mathbb{R}^3 is properly immersed and has an embedded half-open regular neighborhood $N(L)$ on its mean convex side, and $N(L)$ can be chosen to be disjoint from \mathcal{L} if L is not a plane. In particular, if L is a leaf of a weak H -lamination \mathcal{L} of \mathbb{R}^3 , then there is a small perturbation L' of L in $N(L)$ that is properly embedded in \mathbb{R}^3 .

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